جامعة آشور

الكلية التقنية الهندسية

قسم هندسة تقنيات الاتصالات





محاضرات مادة

نمذجة الأنظمة رياضياً

Mathematical Modeling of Systems

المرحلة الثانية

Lecture 3

مدرس المادة

مدرس مساعد كرم ظافر عبد الله

Assistant Lecturer Karam Dhafer Abdullah

COLLEGE OF TECHNICAL ENGINEERING

Department of Communications Engineering Technologies

Mathematical Modeling of Systems

Lecture 3: Homogeneous Second-Order Differential Equations

Second Stage • Lecture Notes

Learning objectives. Recognize homogeneous linear second-order differential equations with constant coefficients; construct their characteristic equations; distinguish three root cases; obtain the complete general solution; and apply initial conditions to determine constants.

1. Introduction

A **second-order differential equation** contains the second derivative of an unknown function $y(x)$. In mathematical modeling, such equations arise in electrical circuits, mechanical vibration and dynamic response of engineering systems.

This lecture focuses on the **linear, homogeneous, constant-coefficient** equation

$$a_0\frac{d^2y}{dx^2} + a_1\frac{dy}{dx} + a_2y = 0, \qquad a_0 \neq 0, \tag{1}$$

where a_0, a_1, a_2 are real constants.

Three essential features
Second order: the highest derivative is y''. **Linear:** y, y', y'' appear to the first power and are not multiplied together. **Homogeneous:** the right-hand side is zero.

2. Operator notation

If $D = \frac{d}{dx}$, then $Dy = y'$ and $D^2y = y''$. For example,

$$y'' + y' + 2y = 0 \quad \Longleftrightarrow \quad (D^2 + D + 2)y = 0.$$

The next step is to turn this differential equation into an algebraic equation.

3. The characteristic equation

Assume an exponential trial solution $y = e^{rx}$, where r is an unknown constant. Then

$$y' = re^{rx}, \qquad y'' = r^2 e^{rx}.$$

Substituting into (1) gives

$$\begin{aligned} a_0 r^2 e^{rx} + a_1 r e^{rx} + a_2 e^{rx} &= 0, \\ e^{rx} \underbrace{(a_0 r^2 + a_1 r + a_2)}_{\text{characteristic polynomial}} &= 0. \end{aligned}$$

Since $e^{rx} \neq 0$, the **characteristic (auxiliary) equation** is

$$\boxed{a_0 r^2 + a_1 r + a_2 = 0.} \tag{2}$$

Its roots determine the type of general solution.

3.1 The discriminant

For a quadratic equation $ar^2 + br + c = 0$, define $\Delta = b^2 - 4ac$. Applied here,

$$\Delta = a_1^2 - 4a_0 a_2, \qquad r_{1,2} = \frac{-a_1 \pm \sqrt{\Delta}}{2a_0}.$$

Condition	Nature of roots	Solution category
$\Delta > 0$	Two distinct real roots	$e^{r_1 x}$ and $e^{r_2 x}$
$\Delta = 0$	One repeated real root	e^{rx} and xe^{rx}
$\Delta < 0$	Two complex-conjugate roots	$e^{\alpha x}\cos\beta x$ and $e^{\alpha x}\sin\beta x$

> **Solution principle.** Because the equation is linear and homogeneous, any linear combination of two linearly independent solutions is also a solution. The constants C_1 and C_2 are arbitrary until two suitable conditions are specified.

4. Standard solution sequence

1. Write the ODE in the form $a_0 y'' + a_1 y' + a_2 y = 0$.
2. Replace y'' by r^2, y' by r and y by 1.
3. Solve the quadratic characteristic equation.
4. Select the correct root case and write the *general solution*.

5. Case I: Two distinct real roots

If $\Delta > 0$, the characteristic equation has two real roots $r_1 \neq r_2$. Two independent solutions are $e^{r_1 x}$ and $e^{r_2 x}$; therefore,

$$\boxed{y(x) = C_1 e^{r_1 x} + C_2 e^{r_2 x}.} \tag{3}$$

Worked example 1: Basic equation from the source

Solve

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = 0.$$

Step 1: Construct the characteristic equation:

$$r^2 + r - 2 = 0.$$

Step 2: Factor and find its roots:

$$(r+2)(r-1) = 0 \quad \Rightarrow \quad r_1 = -2,\ r_2 = 1.$$

Step 3: Since the roots are distinct and real, use (3):

$$\boxed{y = C_1 e^{-2x} + C_2 e^{x}.} \tag{4}$$

Step 4: Check the result by substitution. For e^{-2x}, the left-hand side is

$$(4 - 2 - 2)e^{-2x} = 0;$$

and for e^{x}, it is $(1 + 1 - 2)e^{x} = 0$. Both satisfy the equation.

Worked example 2: Non-unit coefficient

Solve $2y'' - 5y' + 2y = 0$.

$$2r^2 - 5r + 2 = 0 \quad \Rightarrow \quad (2r-1)(r-2) = 0.$$

Thus $r_1 = \frac{1}{2}$ and $r_2 = 2$, so

$$\boxed{y = C_1 e^{x/2} + C_2 e^{2x}.}$$

> **Common error.** Use the coefficient of y'' in the characteristic polynomial. For $2y'' - 5y' + 2y = 0$, the leading term is $2r^2$, *not* r^2.

6. Case II: Repeated real root

If $\Delta = 0$, the characteristic equation has a repeated real root

$$r_1 = r_2 = r = -\frac{a_1}{2a_0}.$$

Using e^{rx} twice would not give two independent solutions. The second solution must be xe^{rx}. Hence,

$$\boxed{y(x) = (C_1 + C_2 x)e^{rx}.} \tag{5}$$

Worked example 3: Repeated positive root

Solve

$$y'' - 4y' + 4y = 0.$$

Step 1: Characteristic equation:

$$r^2 - 4r + 4 = 0.$$

Step 2: Factor:

$$(r-2)^2 = 0 \quad \Rightarrow \quad r_1 = r_2 = 2.$$

Step 3: Write the complete general solution:

$$\boxed{y = (C_1 + C_2 x)e^{2x}.}$$

Worked example 4: Repeated negative root

For $y'' + 6y' + 9y = 0$,

$$r^2 + 6r + 9 = (r+3)^2 = 0 \quad \Rightarrow \quad r = -3.$$

Consequently,

$$\boxed{y = (C_1 + C_2 x)e^{-3x}.}$$

Why does the factor x appear? The functions e^{rx} and xe^{rx} are linearly independent. In fact,

$$\begin{vmatrix} e^{rx} & xe^{rx} \\ re^{rx} & (1+rx)e^{rx} \end{vmatrix} = e^{2rx} \neq 0.$$

This determinant (the Wronskian) confirms that both terms are required in the general solution.

7. Case III: Complex-conjugate roots

If $\Delta < 0$, the two characteristic roots are complex conjugates:

$$r_{1,2} = \alpha \pm i\beta, \qquad \beta > 0,$$

where α is the real part, β is the magnitude of the imaginary part, and $i^2 = -1$. The **real-valued general solution** is

$$\boxed{y = e^{\alpha x}\left(C_1 \cos \beta x + C_2 \sin \beta x\right).} \tag{6}$$

Worked example 5: Complex roots with decay

Solve $y'' + 2y' + 5y = 0$.

$$r^2 + 2r + 5 = 0, \qquad \Delta = 2^2 - 4(1)(5) = -16.$$

Therefore,

$$r = \frac{-2 \pm \sqrt{-16}}{2} = -1 \pm 2i.$$

We identify $\alpha = -1$ and $\beta = 2$. The solution is

$$\boxed{y = e^{-x}\left(C_1 \cos 2x + C_2 \sin 2x\right).}$$

Worked example 6: Pure imaginary roots

Solve $y'' + 4y = 0$.

$$r^2 + 4 = 0 \quad \Rightarrow \quad r = \pm 2i.$$

Here $\alpha = 0$, so $e^{\alpha x} = 1$ and

$$\boxed{y = C_1 \cos 2x + C_2 \sin 2x.}$$

> **Interpretation.** The factor $e^{\alpha x}$ controls the amplitude envelope, while the sine and cosine terms create oscillation. If $\alpha < 0$, the oscillation decreases in amplitude as x increases.

8. Comparing the three root cases

For the ODE $a_0y'' + a_1y' + a_2y = 0$, the characteristic polynomial is always $a_0r^2 + a_1r + a_2$; only the form of the answer changes.

Condition	Roots	General solution
$\Delta > 0$	$r_1 \neq r_2$ (real)	$C_1e^{r_1x} + C_2e^{r_2x}$
$\Delta = 0$	$r_1 = r_2 = r$	$(C_1 + C_2x)e^{rx}$
$\Delta < 0$	$\alpha \pm i\beta$	$e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$

8.1 Classification drill

Identify the roots *before* attempting the complete solution.

1. $y'' - y' - 6y = 0$: $r^2 - r - 6 = (r - 3)(r + 2)$, hence $\Delta > 0$ and the roots are $3, -2$.
2. $y'' + 8y' + 16y = 0$: $(r + 4)^2 = 0$, hence $\Delta = 0$ and $r = -4$ twice.
3. $y'' + 4y' + 8y = 0$: $\Delta = 16 - 32 = -16$, hence $r = -2 \pm 2i$.

8.2 Useful algebraic checks

- If the quadratic factors, factoring is often the fastest method.
- If it does not factor easily, apply the quadratic formula.
- For complex roots, do not omit the imaginary unit i.
- Repeated roots require the factor x in one of the terms.
- A second-order equation normally requires *two* independent constants.

> **Quick self-check.** For $y'' + y' + 2y = 0$, the polynomial $r^2 + r + 2$ has $\Delta = 1 - 8 = -7 < 0$. Thus the answer must involve sine and cosine, *not* two distinct real exponentials.

9. Detailed worked example: $y'' + y' + 2y = 0$

This is the differential equation corresponding to the operator $(D^2 + D + 2)y = 0$ appearing in the supplied source notes.

9.1 Step 1: Construct and solve the quadratic

$$r^2 + r + 2 = 0, \qquad \Delta = 1 - 8 = -7.$$

Using the quadratic formula,

$$r_{1,2} = \frac{-1 \pm \sqrt{-7}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{7}}{2}.$$

Thus $\alpha = -\frac{1}{2}$ and $\beta = \frac{\sqrt{7}}{2}$.

9.2 Step 2: Form the real general solution

$$\boxed{y = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{7}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{7}}{2}x\right)\right].} \tag{7}$$

9.3 Step 3: Verify the method

For the complex root $r = -\frac{1}{2} + i\frac{\sqrt{7}}{2}$,

$$r^2 + r + 2 = 0.$$

Hence e^{rx} solves the ODE. Its real and imaginary parts are the two real independent solutions in (7). Their linear combination also solves the ODE.

10. General versus particular solutions

A **general solution** contains two arbitrary constants C_1, C_2. A **particular solution** results after using two suitable initial conditions, such as

$$y(0) = y_0, \qquad y'(0) = v_0.$$

Initial conditions do not change the characteristic equation; they only determine the two constants.

> **Notation reminder.** The source's quadratic $r^2 + r + 2 = 0$ corresponds to $y'' + y' + 2y = 0$. Keeping the same coefficients in the ODE, operator and characteristic polynomial is essential.

11. Initial-value problems: Real roots

A general solution becomes unique when the two integration constants are determined from two independent conditions.

Worked example 7: Distinct roots and initial data

Solve

$$y'' + y' - 2y = 0, \qquad y(0) = 3, \quad y'(0) = 0.$$

Step 1: From worked example 1,

$$y = C_1 e^{-2x} + C_2 e^{x}.$$

Step 2: Differentiate:

$$y' = -2C_1 e^{-2x} + C_2 e^{x}.$$

Step 3: Apply $x = 0$:

$$C_1 + C_2 = 3, \qquad -2C_1 + C_2 = 0.$$

Subtract the equations (or substitute $C_2 = 2C_1$):

$$3C_1 = 3 \Rightarrow C_1 = 1, \qquad C_2 = 2.$$

Particular solution:

$$\boxed{y = e^{-2x} + 2e^{x}.} \tag{8}$$

Worked example 8: Repeated root and initial data

Solve $y'' - 6y' + 9y = 0$ with $y(0) = 2$ and $y'(0) = 1$.

$$(r-3)^2 = 0 \quad \Rightarrow \quad y = (C_1 + C_2 x)e^{3x}.$$

Differentiate by the product rule:

$$y' = [C_2 + 3(C_1 + C_2 x)]e^{3x}.$$

At $x = 0$, $C_1 = 2$ and $C_2 + 3C_1 = 1$, so $C_2 = -5$. Therefore,

$$\boxed{y = (2 - 5x)e^{3x}.}$$

> **Checking initial conditions.** Always test both $y(0)$ and $y'(0)$ after calculating C_1 and C_2. A correct general form can still lead to a wrong particular solution if a sign is missed.

12. Initial-value problems: Complex roots

Worked example 9: Oscillatory solution with initial data

Solve

$$y'' + 4y' + 13y = 0, \qquad y(0) = 1, \quad y'(0) = 0.$$

Step 1: The characteristic equation is

$$r^2 + 4r + 13 = 0 \quad \Rightarrow \quad r = -2 \pm 3i.$$

Step 2: General solution:

$$y = e^{-2x}(C_1 \cos 3x + C_2 \sin 3x).$$

Step 3: Differentiate carefully using the product rule:

$$\begin{aligned} y' = & -2e^{-2x}(C_1 \cos 3x + C_2 \sin 3x) \\ & + e^{-2x}(-3C_1 \sin 3x + 3C_2 \cos 3x). \end{aligned}$$

Step 4: Apply initial conditions:

$$y(0) = C_1 = 1, \qquad y'(0) = -2C_1 + 3C_2 = 0.$$

Hence $C_2 = \frac{2}{3}$ and the particular solution is

$$\boxed{y = e^{-2x}\left(\cos 3x + \frac{2}{3}\sin 3x\right).} \tag{9}$$

13. Physical meaning of the constants

For $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$:

- α is the **exponential growth or decay rate**.
- β is the **angular frequency** (radians per unit of x).
- C_1, C_2 are fixed by the initial displacement and initial rate.

> **One-minute diagnostic.** If the initial derivative is prescribed, *differentiate the entire product* $e^{\alpha x}(\cdots)$; do not differentiate only the sine and cosine terms.

14. Engineering application: Free vibration

A mass–spring–damper system with no external forcing satisfies

$$\boxed{m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = 0,} \tag{10}$$

where $m > 0$ is mass, $c \geq 0$ is viscous damping, $k > 0$ is spring stiffness and $y(t)$ is displacement. This has exactly the same form as (1), with x replaced by time t.

$$mr^2 + cr + k = 0, \qquad \Delta = c^2 - 4mk.$$

Condition	Physical response
$c^2 > 4mk$	Overdamped: sum of two decaying exponentials.
$c^2 = 4mk$	Critically damped: repeated negative root.
$0 < c^2 < 4mk$	Underdamped: oscillation with a decaying envelope.
$c = 0$	Undamped: sustained sinusoidal vibration.

Worked example 10: Damped vibration

Take $m = 1$, $c = 4$, $k = 13$ (consistent units). Then

$$y'' + 4y' + 13y = 0, \qquad r = -2 \pm 3i.$$

The free response therefore has the form

$$y(t) = e^{-2t}(C_1 \cos 3t + C_2 \sin 3t).$$

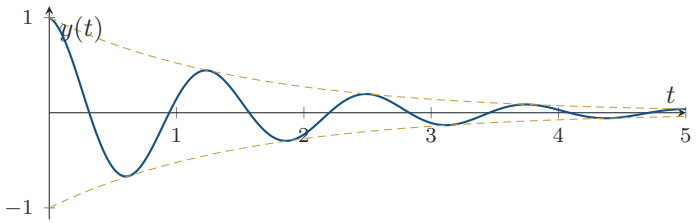

Illustration of a decaying oscillation and its exponential envelopes (schematic).

15. Common mistakes and corrections

1. **Using the same formula for every quadratic.** First determine Δ; repeated and complex roots need different formulas.
2. **Forgetting x with a repeated root.** The correct solution is $(C_1 + C_2x)e^{rx}$, not $(C_1 + C_2)e^{rx}$.
3. **Missing i for negative discriminants.** If $\sqrt{-9} = 3i$, the roots are complex; convert them to a real sine–cosine solution.
4. **Confusing α and β.** In $-2 \pm 3i$, the real part is -2 and the frequency parameter is 3.
5. **Dropping a coefficient.** For $3y'' + 2y' + y = 0$, write $3r^2 + 2r + 1 = 0$ rather than $r^2 + 2r + 1 = 0$.
6. **Stopping at the characteristic roots.** The requested answer is a function $y(x)$, with two constants, unless initial data are provided.

16. Self-assessment exercises

Find the **general solution** to Problems 1–5. For Problem 6, also apply the initial conditions.

1. $y'' - 5y' + 6y = 0$.
2. $y'' + 10y' + 25y = 0$.
3. $y'' + 9y = 0$.
4. $y'' - 2y' + 10y = 0$.
5. $2y'' + y' - y = 0$.
6. $y'' + y' - 2y = 0$, with $y(0) = 1$ and $y'(0) = 4$.

> **Suggested student workflow.** For each equation: identify a_0, a_1, a_2; write $a_0r^2 + a_1r + a_2 = 0$; compute or factor; select the root case; write $y(x)$; then use any initial conditions.

17. Checklist before submitting a solution

- Is the characteristic equation copied with the correct signs?
- Does the final expression have two independent arbitrary constants?
- Are initial conditions applied to both y and y' correctly?
- Does substitution of each basis solution into the ODE give zero?

18. Exercise answers and lecture summary

18.1 Answers to the self-assessment exercises

1. $r^2 - 5r + 6 = (r-2)(r-3)$, so $\boxed{y = C_1 e^{2x} + C_2 e^{3x}}$.
2. $(r+5)^2 = 0$, so $\boxed{y = (C_1 + C_2 x)e^{-5x}}$.
3. $r = \pm 3i$, so $\boxed{y = C_1 \cos 3x + C_2 \sin 3x}$.
4. $r = 1 \pm 3i$, so $\boxed{y = e^x(C_1 \cos 3x + C_2 \sin 3x)}$.
5. $2r^2 + r - 1 = (2r-1)(r+1)$, so $\boxed{y = C_1 e^{x/2} + C_2 e^{-x}}$.
6. $r = -2, 1$, so $y = C_1 e^{-2x} + C_2 e^x$. From $C_1 + C_2 = 1$ and $-2C_1 + C_2 = 4$, obtain $C_1 = -1$, $C_2 = 2$. Therefore, $\boxed{y = -e^{-2x} + 2e^x}$.

19. Key points to remember

1. A linear homogeneous second-order ODE with constant coefficients has the general form $a_0 y'' + a_1 y' + a_2 y = 0$.
2. Use the trial solution e^{rx} to obtain $a_0 r^2 + a_1 r + a_2 = 0$.
3. Distinct real roots produce two exponential terms.
4. A repeated real root produces $(C_1 + C_2 x)e^{rx}$.
5. Complex roots $\alpha \pm i\beta$ produce $e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.
6. Two initial conditions determine the particular solution.

Core source and expansion. The three characteristic-root cases and the example $y'' + y' - 2y = 0$ are drawn from the supplied *Mathematics II*, Section 4.4, pp. 195–196. The additional derivations, examples, initial-value problems, vibration application and exercises were developed to support this lecture.

End of lecture notes

Mathematical Modeling of Systems | Lecture 3