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محاضرات مادة

نمذجة الأنظمة رياضياً

Mathematical Modeling of Systems

المرحلة الثانية

Lecture 1

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Mathematical Modeling of Systems

Lecture 1: Differential Equations

Second Stage • Lecture Notes

Learning objectives. Identify and classify differential equations, distinguish general and particular solutions, solve separable and homogeneous first-order equations.

1. Introduction to differential equations

A **differential equation (DE)** is a mathematical equation involving an unknown function and at least one derivative of that function. The derivative describes how a quantity changes with respect to an independent variable. Common examples from the lecture are

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} = \sin x, \qquad y'' + y' - \ln x = 0. \tag{1}$$

In engineering, a differential equation often relates the behavior of a system to the rate at which its variables change.

2. Classification of differential equations

2.1 Classification by type

Ordinary differential equation (ODE). An ODE contains derivatives with respect to *one* independent variable. For example,

$$\frac{\mathrm{d}y}{\mathrm{d}x} + \sin x = x, \qquad \frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}z}{\mathrm{d}x} = x. \tag{2}$$

Partial differential equation (PDE). A PDE involves a function of two or more independent variables and contains partial derivatives. For example,

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial v}{\partial y} = x, \qquad \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}. \tag{3}$$

How to recognize the type: ordinary derivatives (d) suggest an ODE; partial derivatives (∂) of functions of multiple variables indicate a PDE.

2.2 Classification by order

The **order** of a DE is the order of the highest derivative appearing in it. Compare

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^3 + \sin x = x \qquad \text{first order,} \tag{4}$$

$$\frac{\partial^3 y}{\partial x^3} + \frac{\partial y}{\partial x} = x \qquad \text{third order.} \tag{5}$$

The exponent on a derivative does *not* change its order. Thus, $(y')^3$ still contains only a first derivative.

2.3 Classification by degree

When the equation is polynomial in its derivatives, its **degree** is the exponent of the highest-order derivative, after the equation has been expressed in polynomial form in those derivatives. Examples from the source are

$$(y')^2 + \sin x = x \qquad \text{order 1, degree 2,}$$

$$y'' + y' = \sin x \qquad \text{order 2, degree 1,}$$

$$y''' + y' = \sin x \qquad \text{order 3, degree 1.}$$

2.4 Classification by linearity

A DE is **linear** in the unknown function if the function and its derivatives appear only to the first power, are not multiplied together, and have coefficients that depend only on the independent variable(s). For example,

$$y''' + 5y' + y = 0 \tag{6}$$

is a *linear* third-order ODE. A nonlinear equation may contain terms such as y^2, $(y')^2$, yy', or nonlinear functions of y.

> **Important distinction.** Order and degree describe derivatives; linearity describes *how* the dependent variable and its derivatives enter the equation. They are different classifications.

3. General and particular solutions

A **general solution** contains arbitrary integration constants. A **particular solution** is obtained when suitable initial or boundary conditions determine those constants.

For example, if $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$, integration gives the general solution $y = x^2 + C$. If an initial condition specifies $y(0) = 3$, then $C = 3$ and the particular solution is $y = x^2 + 3$.

4. Separable first-order differential equations

A first-order ODE is **separable** if it can be expressed as

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y) = f(x)g(y), \tag{7}$$

where f depends only on x and g depends only on y. When $g(y) \neq 0$, rearrange and integrate:

$$\frac{1}{g(y)}\,\mathrm{d}y = f(x)\,\mathrm{d}x \qquad \Longrightarrow \qquad \int \frac{1}{g(y)}\,\mathrm{d}y = \int f(x)\,\mathrm{d}x + C. \tag{8}$$

The result may be written implicitly or explicitly for y. When dividing by an expression involving y, check separately whether its zeros also solve the original equation.

4.1 Worked example 1: $\frac{\mathrm{d}y}{\mathrm{d}x} = xy - 4x$

Step 1: Factor the right-hand side:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x(y-4).$$

Step 2: Separate the variables for $y \neq 4$:

$$\frac{1}{y-4}\,\mathrm{d}y = x\,\mathrm{d}x.$$

Step 3: Integrate both sides:

$$\ln|y-4| = \frac{x^2}{2} + C_1.$$

Step 4: Rearrange into explicit form:

$$\boxed{y = 4 + Ce^{x^2/2}.} \tag{9}$$

This family also includes the constant solution $y = 4$ when $C = 0$.

4.2 Worked example 2: $\frac{\mathrm{d}y}{\mathrm{d}x} = 3y^2 - y^2\sin x$

Factor the equation:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y^2(3 - \sin x).$$

For $y \neq 0$, separate variables and integrate:

$$\begin{aligned} \frac{1}{y^2}\,\mathrm{d}y &= (3-\sin x)\,\mathrm{d}x, \\ \int y^{-2}\,\mathrm{d}y &= \int (3-\sin x)\,\mathrm{d}x, \\ -\frac{1}{y} &= 3x + \cos x + C_1. \end{aligned}$$

Hence,

$$\boxed{y = \frac{1}{C - 3x - \cos x}.} \tag{10}$$

> **Additional check:** $y = 0$ also satisfies the original equation. It is excluded while dividing by y^2, so it should be recorded separately.

4.3 Worked example 3: $\frac{\mathrm{d}y}{\mathrm{d}x} = xy - 3x - 2y + 6$

Group terms to factor the right-hand side:

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= x(y-3) - 2(y-3) \\ &= (x-2)(y-3). \end{aligned}$$

Separate the variables and integrate:

$$\frac{1}{y-3}\,\mathrm{d}y = (x-2)\,\mathrm{d}x,$$

$$\ln|y-3| = \frac{x^2}{2} - 2x + C_1.$$

Thus,

$$\boxed{y = 3 + Ce^{\frac{x^2}{2}-2x}.} \tag{11}$$

The constant solution $y = 3$ is obtained when $C = 0$.

5. Homogeneous first-order differential equations

5.1 Homogeneous functions

A function $f(x, y)$ is **homogeneous of degree** n if, for admissible scaling factors $t > 0$,

$$f(tx, ty) = t^n f(x, y). \tag{12}$$

In particular, a function of degree zero satisfies $f(tx, ty) = f(x, y)$ and depends on a ratio such as y/x or x/y (on an appropriate domain).

Example (degree 4). Consider

$$f(x, y) = 2y^4 - x^2y^2.$$

Then

$$f(tx, ty) = 2t^4y^4 - t^4x^2y^2 = t^4 f(x, y),$$

so f has degree 4.

Example (degree 0). Expressions such as y/x, $e^{x/y}$, and $\sin(x/y)$ do not change when x and y are both multiplied by the same nonzero factor. Consequently, a sum of such expressions, like

$$f(x, y) = \frac{y}{x} - 3e^{x/y} + \sin\left(\frac{x}{y}\right),$$

is homogeneous of degree zero where it is defined.

5.2 Homogeneous differential equations

A first-order DE of the form

$$\frac{\mathrm{d}y}{\mathrm{d}x} = F(x, y) \tag{13}$$

is **homogeneous** when F is homogeneous of degree zero. The same idea can be expressed in differential form:

$$P(x, y)\,\mathrm{d}x + Q(x, y)\,\mathrm{d}y = 0, \tag{14}$$

where P and Q are homogeneous functions of the *same degree*.

Examples reproduced from the source can be classified by comparing the degrees of the coefficients:

- $x(y + x)\,\mathrm{d}x = y^2\,\mathrm{d}y$ is homogeneous of degree 2.
- $(xy + x)\,\mathrm{d}x = (y^2 - x^2)\,\mathrm{d}y$ is not homogeneous because the terms in the first coefficient do not share one degree.
- $y(\ln y - \ln x - 1)\,\mathrm{d}x + x\,\mathrm{d}y = 0$ is homogeneous of degree 1, where the logarithms are defined, because $\ln(ty) - \ln(tx) = \ln y - \ln x$ for $t > 0$.
- $y(\ln y - 1)\,\mathrm{d}x + x\,\mathrm{d}y = 0$ is not homogeneous.

6. Converting a homogeneous equation into separable form

A homogeneous first-order equation can be solved with the substitution

$$\boxed{v = \frac{y}{x}, \qquad y = vx.} \tag{15}$$

Because v depends on x, differentiate using the product rule:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}. \tag{16}$$

When $F(x, y) = \phi(y/x)$, Equation (13) becomes

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= \phi(v), \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= \phi(v) - v, \\ \frac{\mathrm{d}v}{\phi(v) - v} &= \frac{\mathrm{d}x}{x}. \end{aligned}$$

The last expression is separable on intervals with $x \neq 0$ and $\phi(v) - v \neq 0$. After integration, substitute $v = y/x$ to express the answer in terms of the original variables. Before dividing by $\phi(v) - v$, check whether any constant $v = k$ satisfies $\phi(k) = k$: these give additional straight-line solutions $y = kx$. An alternative substitution $x = vy$ may be useful.

6.1 Worked example 4: trigonometric homogeneous equation

Solve

$$x\cos\left(\frac{y}{x}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = y\cos\left(\frac{y}{x}\right) + x. \tag{17}$$

Rearranging shows that its right-hand side is a function of y/x:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y}{x} + \frac{1}{\cos(y/x)}.$$

Set $y = vx$, which gives $\frac{\mathrm{d}y}{\mathrm{d}x} = v + x\frac{\mathrm{d}v}{\mathrm{d}x}$. Substitution into Equation (17) yields

$$\begin{aligned} v + x\frac{\mathrm{d}v}{\mathrm{d}x} &= v + \frac{1}{\cos v}, \\ x\frac{\mathrm{d}v}{\mathrm{d}x} &= \frac{1}{\cos v}, \\ \cos v\,\mathrm{d}v &= \frac{\mathrm{d}x}{x}. \end{aligned}$$

Integrating and substituting back gives the implicit general solution

$$\boxed{\sin\left(\frac{y}{x}\right) = \ln|x| + C.} \tag{18}$$

6.2 Worked example 5: polynomial homogeneous equation

Solve

$$2(2x^2 + y^2)\,\mathrm{d}x - xy\,\mathrm{d}y = 0. \tag{19}$$

Both coefficients are homogeneous of degree 2. Using $y = vx$ and $\mathrm{d}y = v\,\mathrm{d}x + x\,\mathrm{d}v$, Equation (19) becomes

$$2(2x^2 + v^2x^2)\,\mathrm{d}x - vx^2(v\,\mathrm{d}x + x\,\mathrm{d}v) = 0,$$
$$x^2(4 + v^2)\,\mathrm{d}x - vx^3\,\mathrm{d}v = 0.$$

For $x \neq 0$, divide and separate:

$$\frac{\mathrm{d}x}{x} = \frac{v\,\mathrm{d}v}{4 + v^2}.$$

Integration gives

$$\ln|x| = \frac{1}{2}\ln(4 + v^2) + C_1.$$

Using $v = y/x$ and consolidating the integration constants produces

$$\boxed{x^4 = C(4x^2 + y^2).} \tag{20}$$

7. Key points to remember

1. Identify whether a DE is ordinary or partial, then determine its order, degree (when defined), and linearity.
2. Distinguish a general solution from a particular solution obtained from conditions.
3. For separable equations, collect all y terms with $\mathrm{d}y$ and all x terms with $\mathrm{d}x$ before integrating.
4. For homogeneous first-order equations, use $y = vx$ and the product rule $y' = v + xv'$; check solutions excluded by division.

End of lecture notes