Differential Mathematics

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Lecture 7

Theory and rules of derivatives

Derivatives(Theory and rules of derivatives)

The derivative is one of the key ideas in calculus, and is used to study a wide range of problems in mathematics, science, economics, and medicine.

Tangents and the Derivative at a Point

In this section we define the slope and tangent to a curve at a point, and the derivative of a function at a point. The derivative gives a way to find both the slope of a graph and the instantaneous rate of change of a function.

Finding a Tangent to the Graph of a Function

To find a tangent to an arbitrary curve $y = f(x)$ at a point $P(x_0, f(x_0))$, we use the procedure introduced in Section 2.1. We calculate the slope of the secant through P and a nearby point $Q(x_0 + h, f(x_0 + h))$. We then investigate the limit of the slope as $h \to 0$ (Figure 3.1). If the limit exists, we call it the slope of the curve at P and define the tangent at P to be the line through P having this slope.

DEFINITIONS The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$m = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \qquad \text{(provided the limit exists).}$$

The **tangent line** to the curve at P is the line through P with this slope.

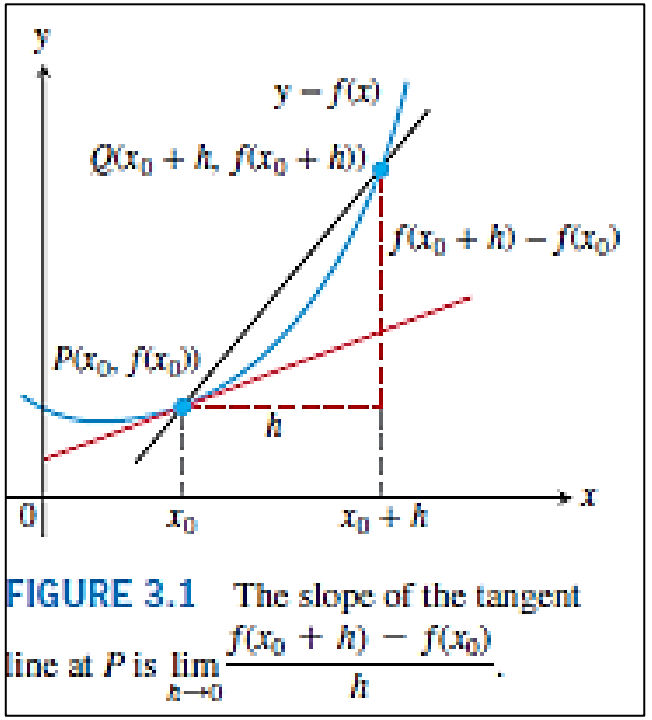


FIGURE 3.1 The slope of the tangent line at P is $\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

In Section 2.1, Example 3, we applied these definitions to find the slope of the parabola $f(x) = x^2$ at the point $P(2, 4)$ and the tangent line to the parabola at P. Let's look at another example.

EXAMPLE 1

(a) Find the slope of the curve $y = 1/x$ at any point $x = a \neq 0$. What is the slope at the point $x = -1$?

(b) Where does the slope equal -1/4?

(c) What happens to the tangent to the curve at the point $(a, 1/a)$ as a changes?

Solution

(a) Here $f(x) = 1/x$. The slope at $(a, 1/a)$ is

$$\lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{1}{a+h}-\frac{1}{a}}{h} = \lim_{h\to 0}\frac{1}{h}\frac{a-(a+h)}{a(a+h)}$$

$$= \lim_{h\to 0}\frac{-h}{ha(a+h)} = \lim_{h\to 0}\frac{-1}{a(a+h)} = -\frac{1}{a^2}.$$

Notice how we had to keep writing "$\lim_{h\to 0}$" before each fraction until the stage at which we could evaluate the limit by substituting $h = 0$. The number a may be positive or negative, but not 0. When $a = -1$, the slope is $-1/(-1)^2 = -1$ (Figure 3.2).

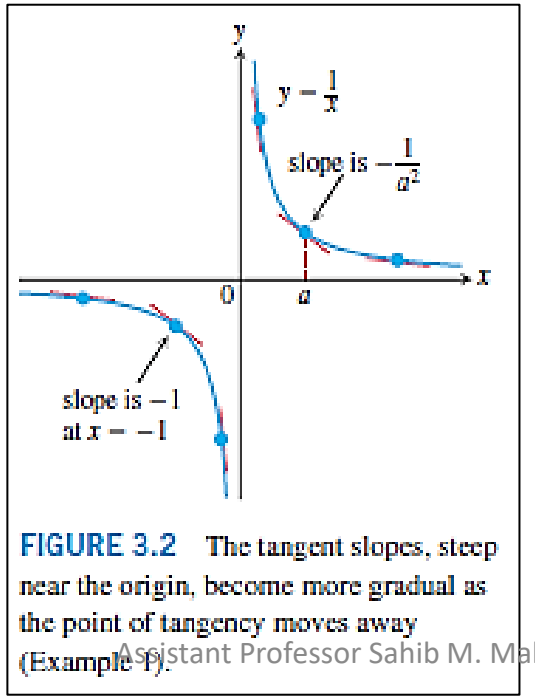


FIGURE 3.2 The tangent slopes, steep near the origin, become more gradual as the point of tangency moves away (Example 1).

(b) The slope of $y = 1/x$ at the point where $x = a$ is $-1/a^2$. It will be -1/4 provided that

$$-\frac{1}{a^2} = -\frac{1}{4}.$$

This equation is equivalent to $a^2 = 4$, so $a = 2$ or $a = -2$. The curve has slope -1/4 at the two points (2, 1/2) and (-2, -1/2) (Figure 3.3).

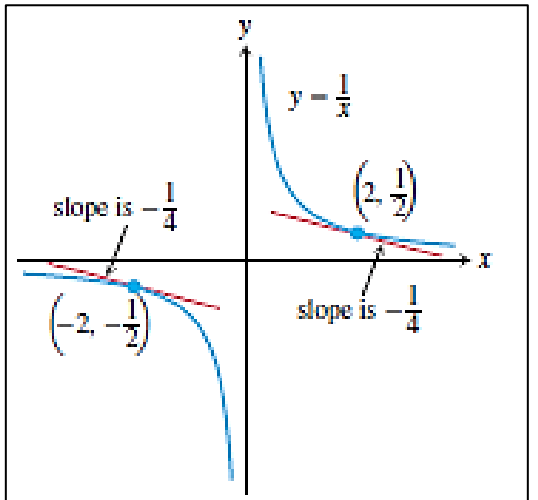


FIGURE 3.3 The two tangent lines to $y = 1/x$ having slope $-1/4$ (Example 1).

(c) The slope $-1/a^2$ is always negative if $a \neq 0$. As $a \to 0^+$, the slope approaches $-\infty$ and the tangent becomes increasingly steep (Figure 3.2). We see this situation again as $a \to 0^-$. As a moves away from the origin in either direction, the slope approaches 0 and the tangent levels off becoming more and more horizontal.

Rates of Change: Derivative at a Point

The expression

$$\frac{f(x_0 + h) - f(x_0)}{h}, \quad h \neq 0$$

is called the **difference quotient of f at x_0 with increment h**. If the difference quotient has a limit as h approaches zero, that limit is given a special name and notation.

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

The notation $f'(x0)$ is read "f prime of $x0$."

provided this limit exists.

If we interpret the difference quotient as the slope of a secant line, then the derivative gives the slope of the curve $y = f(x)$ at the point $P(x_0, f(x_0))$. Exercise 33 shows that the derivative of the linear function $f(x) = mx + b$ at any point x_0 is simply the slope of the line, so

$$f'(x_0) = m,$$

which is consistent with our definition of slope.

If we interpret the difference quotient as an average rate of change (Section 2.1), the derivative gives the function's instantaneous rate of change with respect to x at the point $x = x_0$. We study this interpretation in Section 3.4.

EXAMPLE 2 In Examples 1 and 2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. We knew that the rock fell $y = 16t^2$ feet during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's *exact* speed at this time?

Solution We let $f(t) = 16t^2$. The average speed of the rock over the interval between $t = 1$ and $t = 1 + h$ seconds, for $h > 0$, was found to be

$$\frac{f(1 + h) - f(1)}{h} = \frac{16(1 + h)^2 - 16(1)^2}{h} = \frac{16(h^2 + 2h)}{h} = 16(h + 2).$$

The rock's speed at the instant $t = 1$ is then

$$f'(1) = \lim_{h \to 0} 16(h + 2) = 16(0 + 2) = 32 \text{ ft/sec}.$$

Our original estimate of 32 ft/sec in Section 2.1 was right.

The following are all interpretations for the limit of the difference quotient

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

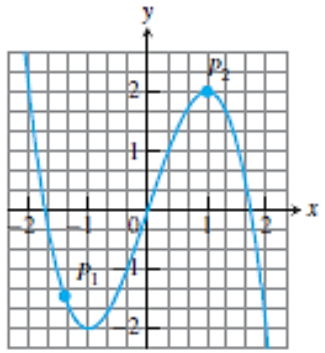
1. The slope of the graph of $y = f(x)$ at $x = x_0$
2. The slope of the tangent to the curve $y = f(x)$ at $x = x_0$
3. The rate of change of $f(x)$ with respect to x at $x = x_0$
4. The derivative $f'(x_0)$ at a point

Home work 10

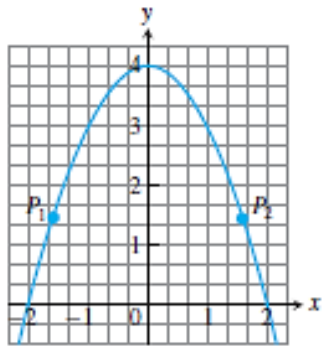
Slopes and Tangent Lines

In Exercises 1–2, use the grid and a straight edge to make a rough estimate of the slope of the curve (in y-units per x-unit) at the points $P1$ and $P2$.

1-



2-



In Exercises 3–5, find an equation for the tangent to the curve at the given point. Then sketch the curve and tangent together.

3. $y = 4 - x^2$, $(-1, 3)$

4. $y = (x - 1)^2 + 1$, $(1, 1)$

5. $y = \frac{1}{x^3}$, $\left(-2, -\frac{1}{8}\right)$

In Exercises 6–9, find the slope of the function's graph at the given point. Then find an equation for the line tangent to the graph there.

6. $f(x) = x^2 + 1$, $(2, 5)$

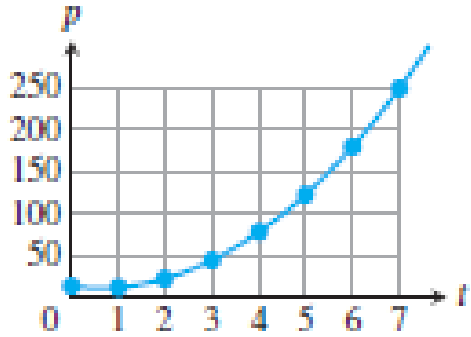
7. $h(t) = t^3 + 3t$, $(1, 4)$

8. $g(x) = \frac{x}{x - 2}$, $(3, 3)$

9. $f(x) = \sqrt{x + 1}$, $(8, 3)$

Interpreting Derivative Values

10. Growth of yeast cells In a controlled laboratory experiment, yeast cells are grown in an automated cell culture system that counts the number P of cells present at hourly intervals. The number after t hours is shown in the accompanying figure.



a. Explain what is meant by the derivative $P'(5)$. What are its units?

b. Which is larger, $P'(2)$ or $P'(3)$? Give a reason for your answer.

c. The quadratic curve capturing the trend of the data points (see Section 1.4) is given by $P(t) = 6.10t^2 - 9.28t + 16.43$. Find the instantaneous rate of growth when $t = 5$ hours.

The Derivative as a Function

In the last section we defined the derivative of $y = f(x)$ at the point $x = x_0$ to be the limit

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We now investigate the derivative as a *function* derived from f by considering the limit at each point x in the domain of f.

DEFINITION The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

We use the notation $f(x)$ in the definition to emphasize the independent variable x with respect to which the derivative function $f'(x)$ is being defined. The domain of f' is the set of points in the domain of f for which the limit exists, which means that the domain may be the same as or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative) at x**. If f' exists at every point in the domain of f, we call f **differentiable**.If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.4). This formula is sometimes more convenient to use when finding a derivative function, and focuses on the point z that approaches x.

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

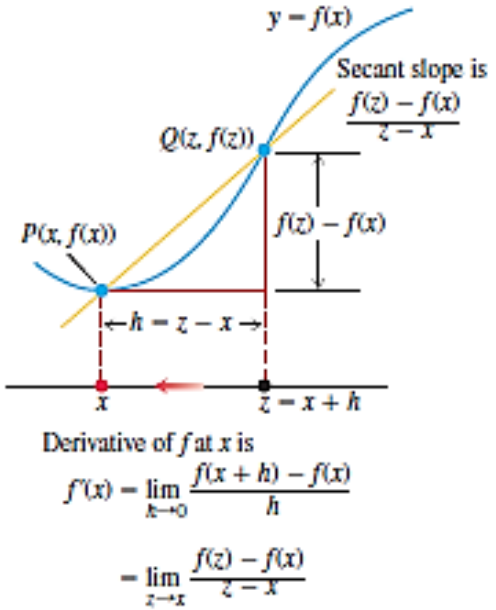


FIGURE 3.4 Two forms for the difference quotient.

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Example 1 of Section 3.1 illustrated the differentiation process for the function $y = 1/x$ when $x = a$. For x representing any point in the domain, we get the formula

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Derivative of the Reciprocal Function

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad x \neq 0$$

Here are two more examples in which we allow x to be any point in the domain of f.

EXAMPLE 1 Differentiate $f(x) = \dfrac{x}{x-1}$.

Solution We use the definition of derivative, which requires us to calculate $f(x+h)$ and then subtract $f(x)$ to obtain the numerator in the difference quotient. We have

$$f(x) = \frac{x}{x-1} \quad \text{and} \quad f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \qquad \text{Definition}$$

$$= \lim_{h\to 0} \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad-cb}{bd}$$

$$= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \qquad \text{Simplify.}$$

$$= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \qquad \text{Cancel } h \neq 0.$$

EXAMPLE 2

(a) Find the derivative of $f(x) = \sqrt{x}$ for $x < 0$.
(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

Solution

(a) We use the alternative formula to calculate f':

$$f'(x) = \lim_{z\to x} \frac{f(z)-f(x)}{z-x}$$

$$= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{z-x}$$

$$= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{(\sqrt{z}-\sqrt{x})(\sqrt{z}+\sqrt{x})}$$

$$= \lim_{z\to x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

(b) The slope of the curve at $x = 4$ is $f'(4) = \dfrac{1}{2\sqrt{4}} = \dfrac{1}{4}$.

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.5):

$$y = 2 + \frac{1}{4}(x-4)$$

$$y = \frac{1}{4}x + 1.$$

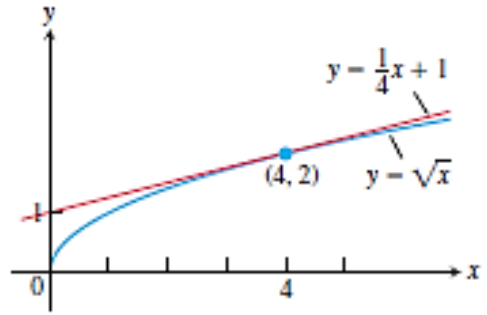


FIGURE 3.5 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

Derivative of the Square Root Function

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}, \quad x > 0$$

The symbols d/dx and D indicate the operation of differentiation. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio (until we introduce the idea of "differentials" in Section 3.11). To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

For instance, in Example 2

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

Graphing the Derivative

We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f. That is, we plot the points $(x, f'(x))$ in the xy-plane and connect them with a smooth curve, which represents $y = f'(x)$.

EXAMPLE 3 Graph the derivative of the function $y = f(x)$ in Figure 3.6a.

Solution We sketch the tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a smooth curve as sketched in Figure 3.6b.

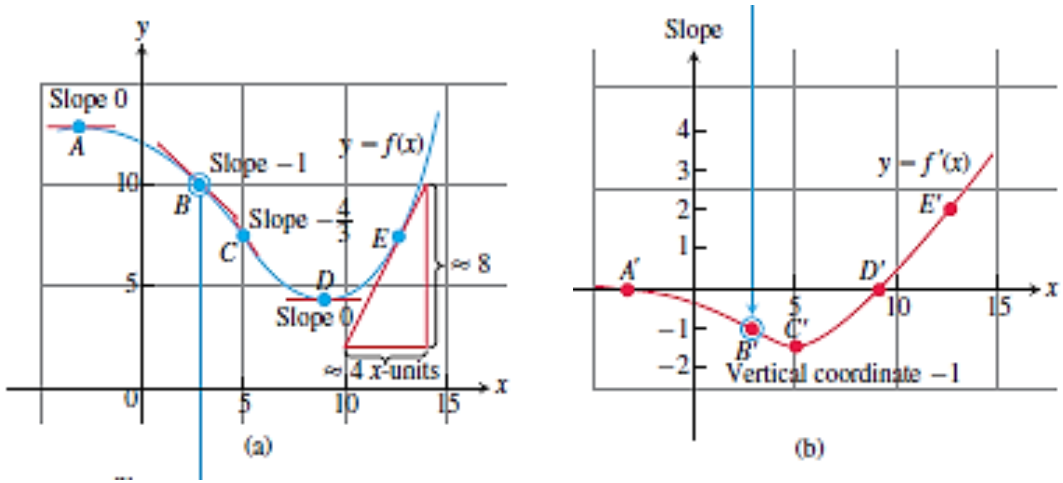


FIGURE 3.6 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B and soon. The slope at E is approximately $8/4 = 2$. In (b) we see that the rate of change of f is negative for x between A' and D'; the rate of change is positive for x to the right of D'.

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7).

Right-hand and left-hand derivatives may be defined at any point of a function's domain. Because of Theorem 6, Section 2.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

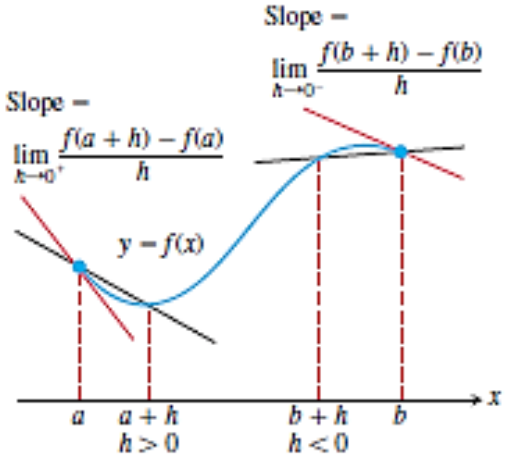


FIGURE 3.7 Derivatives at endpoints of a closed interval are one-sided limits.

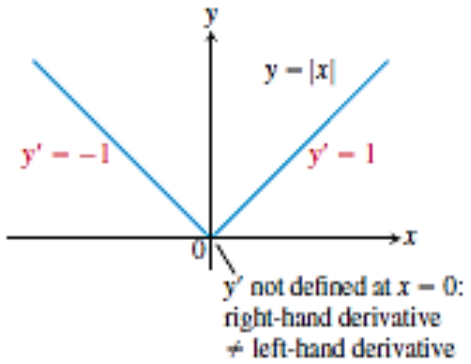


FIGURE 3.8 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner" (Example 4).

EXAMPLE 4 Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution From Section 3.1, the derivative of $y = mx + b$ is the slope m. Thus, to the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m,\ |x| = x$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x$$

$$\text{Right-hand derivative of } |x| \text{ at zero} = \lim_{h \to 0^+} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^+} \frac{|h|}{h}$$

$$= \lim_{h \to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0$$

$$= \lim_{h \to 0^+} 1 = 1$$

$$\text{Left-hand derivative of } |x| \text{ at zero} = \lim_{h \to 0^-} \frac{|0 + h| - |0|}{h} = \lim_{h \to 0^-} \frac{|h|}{h}$$

$$= \lim_{h \to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0$$

$$= \lim_{h \to 0^-} -1 = -1.$$

EXAMPLE 5 In Example 2 we found that for $x < 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach ∞, the graph has a *vertical tangent* at the origin. (See Figure 1.17 on page 9.)

Home work 11

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2$; $f'(-3), f'(0), f'(1)$
2. $F(x) = (x-1)^2 + 1$; $F'(-1), F'(0), F'(2)$
3. $g(t) = \frac{1}{t^2}$; $g'(-1), g'(2), g'(\sqrt{3})$
4. $k(z) = \frac{1-z}{2z}$; $k'(-1), k'(1), k'(\sqrt{2})$

In Exercises 5–8, find the indicated derivatives.

5. $\frac{dy}{dx}$ if $y = 2x^3$
6. $\frac{ds}{dt}$ if $s = \frac{t}{2t+1}$
7. $\frac{dp}{dq}$ if $p = q^{3/2}$
8. $\frac{dz}{dw}$ if $z = \frac{1}{\sqrt{w^2-1}}$

Slopes and Tangent Lines

9. Differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

$$y = \frac{x+3}{1-x}, \quad x = -2$$

10. Differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function

$$w = g(z) = 1 + \sqrt{4-z}, \quad (z, w) = (3, 2)$$

11. Find the values of the derivatives.

$$\left.\frac{dr}{d\theta}\right|_{\theta=0} \text{ if } r = \frac{2}{\sqrt{4-\theta}}$$

12. Use the formula $f'(x) = \lim_{z\to x} \frac{f(z) - f(x)}{z - x}$

to find the derivative of the function $g(x) = 1 + \sqrt{x}$

List of Derivative Rules

1 - Constant Rules: $f(x) = C$ then $f'(x) = 0$

2 – Power Rule : $f(x) = x^n$ then $f'(x) = n\,x^{n-1}$

3 - Constant Multiple Rule : $g(x) = C \cdot f(x)$ then $g'(x) = C \cdot f'(x)$

4 – Sum and Difference Rule : $h(x) = f(x) \pm g(x)$ then $h'(x) = f'(x) \pm g'(x)$

5 – Product Rule : $h(x) = f(x)\,g(x)$ then $h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

6 – Quotient Rule : $h(x) = \dfrac{f(x)}{g(x)}$ then $h'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$

7 - Chain Rule : : $h(x) = f(g(x))$ then $h'(x) = f'(g(x))\,g'(x)$

1- Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}, \quad \text{or} \quad \frac{d}{dx}(x^{-1}) = -x^{-2}.$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}, \quad \text{or} \quad \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2}.$$

2- Power Rule (General Version)

If n is any real number, then

$$\frac{d}{dx}x^n = nx^{n-1},$$

EXAMPLE 6 Differentiate the following powers of x.

(a) x^3 (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

Solution:

Applying the Power Rule
Subtract 1 from the exponent and multiply the result by the original exponent.

(a) $\dfrac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$

(b) $\dfrac{d}{dx}(x^{2/3}) = \dfrac{2}{3}x^{(2/3)-1} = \dfrac{2}{3}x^{-1/3}$

(c) $\dfrac{d}{dx}(x^{\sqrt{2}}) = \sqrt{2}x^{\sqrt{2}-1}$

(d) $\dfrac{d}{dx}\left(\dfrac{1}{x^4}\right) = \dfrac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\dfrac{4}{x^5}$

(e) $\dfrac{d}{dx}(x^{-4/3}) = -\dfrac{4}{3}x^{-(4/3)-1} = -\dfrac{4}{3}x^{-7/3}$

(f) $\dfrac{d}{dx}(\sqrt{x^{2+\pi}}) = \dfrac{d}{dx}(x^{1+(\pi/2)}) = \left(1 + \dfrac{\pi}{2}\right)x^{1+(\pi/2)-1} = \dfrac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$

The next rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

3 - Derivative Constant Multiple Rule

EXAMPLE 7

(a) The derivative formula $\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.10).

(b) Negative of a function

The derivative of the negative of a differentiable function u is the negative of the function's derivative. The Constant Multiple Rule with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives

Denoting Functions by u and v

The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. We do not want to use these same letters when stating general differentiation rules, so instead we use letters like u and v that are not likely to be already in use.

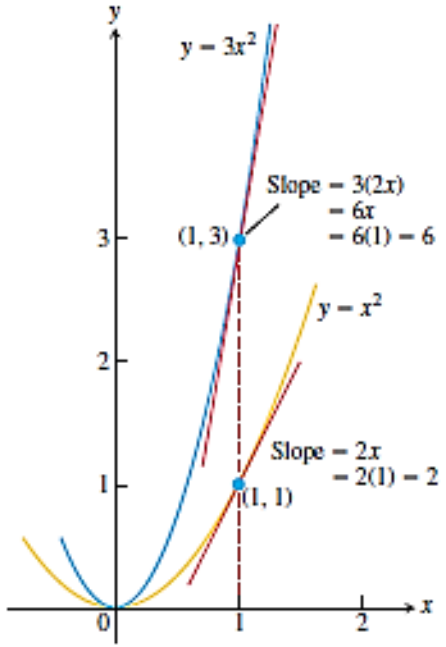


FIGURE 3.10 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinate triples the slope (Example 2).

4 - Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

For example, if $y = x^4 + 12x$, then y is the sum of $u(x) = x^4$ and $v(x) = 12x$. We then have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) = 4x^3 + 12.$$

EXAMPLE 8 Find the derivative of the polynomial

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1.$$

Solution:

$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$

Sum and Difference Rules

$$= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 = 3x^2 + \frac{8}{3}x - 5$$

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EXAMPLE 9 Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so,where?

Solution: The horizontal tangents, if any, occur where the slope dy/dx is zero. We have

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$4x^3 - 4x = 0$$
$$4x(x^2 - 1) = 0$$
$$x = 0, 1, -1.$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0$, 1, and -1. The corresponding points on the curve are (0, 2), (1,1), and (-1, 1). See Figure 3.11.

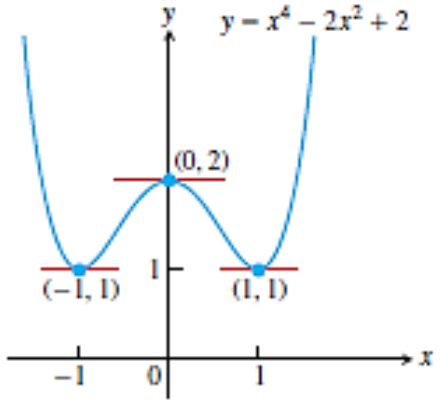


FIGURE 3.11 The curve in Example 4 and its horizontal tangents.

5 - Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

EXAMPLE 10 Find an equation for a line that is tangent to the graph of $y = e^x$ and goes through the origin.

Solution Since the line passes through the origin, its equation is of the form $y = mx$, where m is the slope. If it is tangent to the graph at the point (a, e^a), the slope is $m = (e^a - 0)/(a - 0)$. The slope of the natural exponential at $x = a$ is e^a. Because these slopes are the same, we then have that $e^a = e^a/a$. It follows that $a = 1$ and $m = e$, so the equation of the tangent line is $y = e^x$. See Figure 3.13.

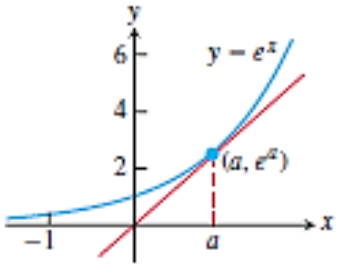


FIGURE 3.13 The line through the origin is tangent to the graph of $y = e^x$ when $a = 1$ (Example 5).

6 - Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Equation (3) is equivalent to saying that

$$(fg)' = f'g + fg'.$$

This form of the Product Rule is useful and applies to *dot products* and *cross products* of *vector-valued* functions, studied in Chapter 13.

EXAMPLE 11 Find the derivative of

(a) $y = \frac{1}{x}(x^2 + e^x)$, **(b)** $y = e^{2x}$.

Solution

(a) We apply the Product Rule with $u = 1/x$ and $v = x^2 + e^x$:

$$\frac{d}{dx}\left[\frac{1}{x}(x^2 + e^x)\right] = \frac{1}{x}(2x + e^x) + (x^2 + e^x)\left(-\frac{1}{x^2}\right)$$

$$= 2 + \frac{e^x}{x} - 1 - \frac{e^x}{x^2}$$

$$= 1 + (x - 1)\frac{e^x}{x^2}.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$

(b) $\frac{d}{dx}(e^{2x}) = \frac{d}{dx}(e^x \cdot e^x) = e^x \cdot \frac{d}{dx}(e^x) + e^x \cdot \frac{d}{dx}(e^x) = 2e^x \cdot e^x = 2e^{2x}$

EXAMPLE 12 Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\frac{d}{dx}\left[(x^2 + 1)(x^3 + 3)\right] = (x^2 + 1)(3x^2) + (x^3 + 3)(2x)$$

$$= 3x^4 + 3x^2 + 2x^4 + 6x$$

$$= 5x^4 + 3x^2 + 6x.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$y = (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3$$

$$\frac{dy}{dx} = 5x^4 + 3x^2 + 6x.$$

7 - Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

EXAMPLE 12 Find the derivative of **(a)** $y = \frac{t^2 - 1}{t^3 + 1}$, **b)** $y = e^{-x}$.

Solution

(a) We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^3 + 1$:

$$\frac{dy}{dt} = \frac{(t^3 + 1)\cdot 2t - (t^2 - 1)\cdot 3t^2}{(t^3 + 1)^2}$$

$$= \frac{2t^4 + 2t - 3t^4 + 3t^2}{(t^3 + 1)^2}$$

$$= \frac{-t^4 + 3t^2 + 2t}{(t^3 + 1)^2}.$$

$\frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$

(b) $\frac{d}{dx}(e^{-x}) = \frac{d}{dx}\left(\frac{1}{e^x}\right) = \frac{e^x \cdot 0 - 1 \cdot e^x}{(e^x)^2} = \frac{-1}{e^x} = -e^{-x}$

EXAMPLE 13 Find the derivative of

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4}.$$

Solution Using the Quotient Rule here will result in a complicated expression with many terms. Instead, use some algebra to simplify the expression. First expand the numerator and divide by x^4:

$$y = \frac{(x - 1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$

$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''.
So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. It is written in several ways:

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}(6x^5) = 30x^4.$$

Thus $D^2(x^6) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$, is the **third derivative** of y with respect to x. The names continue as you imagine, with

EXAMPLE 14 The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$
Second derivative: $y'' = 6x - 6$
Third derivative: $y''' = 6$
Fourth derivative: $y(4) = 0$.
All polynomial functions have derivatives of all orders. In this example, the fifth and later derivatives are all zero.

How to Read the Symbols for Derivatives

y'	"y prime"
y''	"y double prime"
$\frac{d^2y}{dx^2}$	"d squared y dx squared"
y'''	"y triple prime"
$y^{(n)}$	"y super n"
$\frac{d^ny}{dx^n}$	"d to the n of y by dx to the n"
D^n	"D to the n"

Home work 12

Derivative Calculations

In Exercises 1–6, find the first and second derivatives.

1. $y = -x^2 + 3$ **2.** $y = x^2 + x + 8$ **3.** $s = 5t^3 - 3t^5$

4. $w = 3z^7 - 7z^3 + 21z^2$. **5.** $r = \frac{1}{3s^2} - \frac{5}{2s}$ **6.** $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

Find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

7. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 8–12.

8. $y = \frac{2x + 5}{3x - 2}$ **9.** $g(x) = \frac{x^2 - 4}{x + 0.5}$ **10.** $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$

11. $y = x^{-3/5} + \pi^{3/2}$ **12.** $r = e^{\theta}\left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right)$

13. Find the first and second derivatives of the function

$$p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$$

14. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a. $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

15. Find all points (x, y) on the graph of $f(x) = 3x^2 - 4x$ with tangent lines parallel to the line $y = 8x + 5$.

16. Find the value of a that makes the following function differentiable for all x-values.

$$g(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x, & \text{if } x \geq 0 \end{cases}$$