Differential Mathematics

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Lecture 3

Transcendental functions-Hyperbolic and inverse hyperbolic functions

Hyperbolic Functions

In this section, we will learn about:

Hyperbolic functions and Inverse Hyperbolic

Hyperbolic Functions:

The hyperbolic functions are formed by taking combinations of the two exponential functions

To denote the hyperbolic versions, an **h** is appended to their abbreviations and the name by which they are called pronounced with some obvious "h". This is done in different ways, but the idea is simply to briefly convey that the given function is a hyperbolic version of a corresponding trigonometric one. After all "hyperbolic cosecant" is a bit of a mouthful.

"hyperbolic sine" (**sinh**), called "shine" or "cinch"

"hyperbolic cosine" (**cosh**), called "cosh"

"hyperbolic tangent" (**tanh**), called "tanch"

"hyperbolic secant" (**sech**), called "shek" or "sech"

"hyperbolic cosecant" (**csch** or **cosech**), called "coshek" or "cosech"

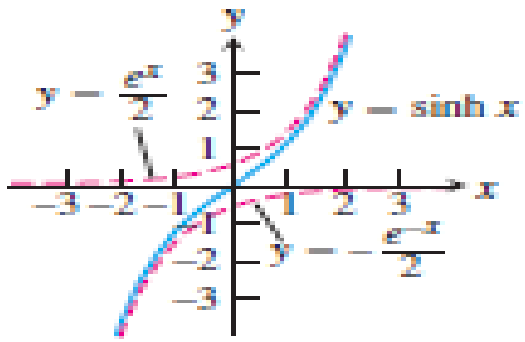
"hyperbolic cotangent" (**coth**), called "coth"

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csc\,h} x = \frac{1}{\sinh x}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sec\,h} x = \frac{1}{\cosh x}$$

$$\tanh x = \frac{\sinh x}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}$$

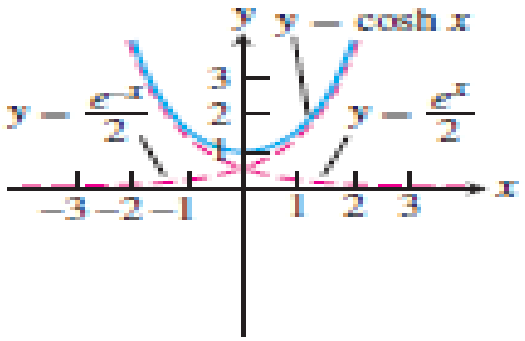
The six basic hyperbolic functions



(a)

Hyperbolic sine:

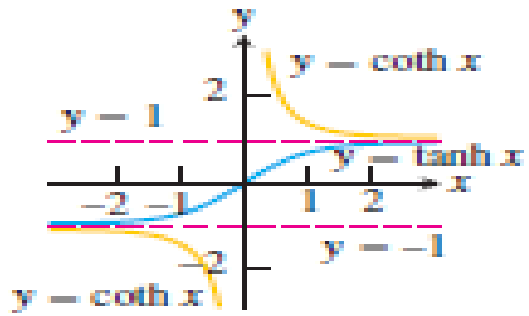
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



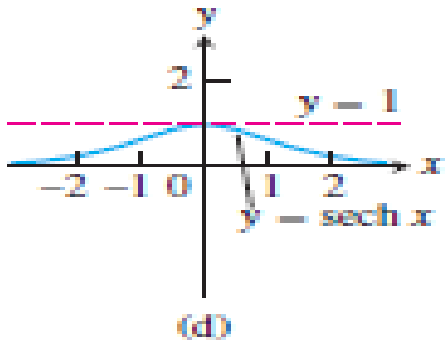
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

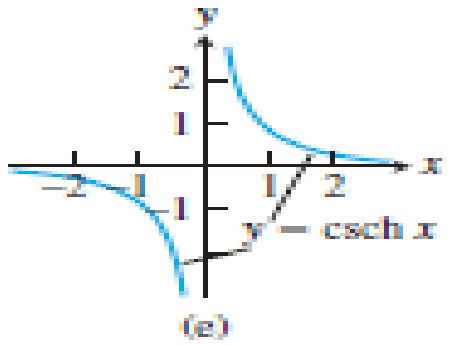
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

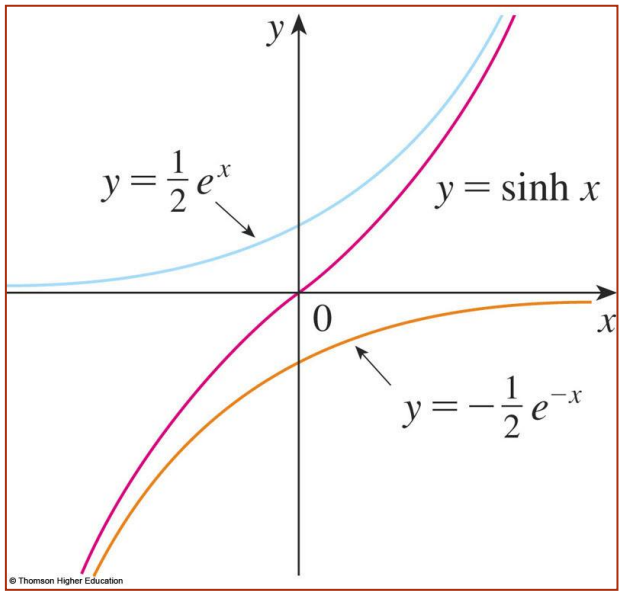


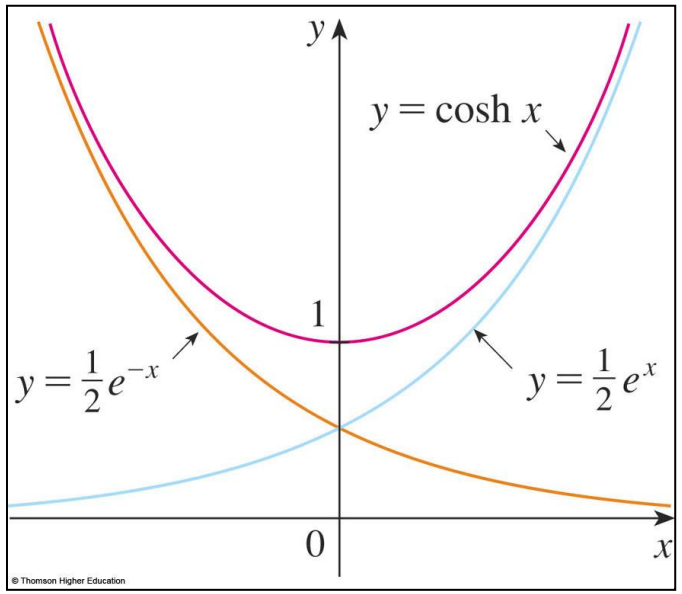
(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

The graphs of hyperbolic sine and cosine can be sketched using graphical addition, as in these figures.

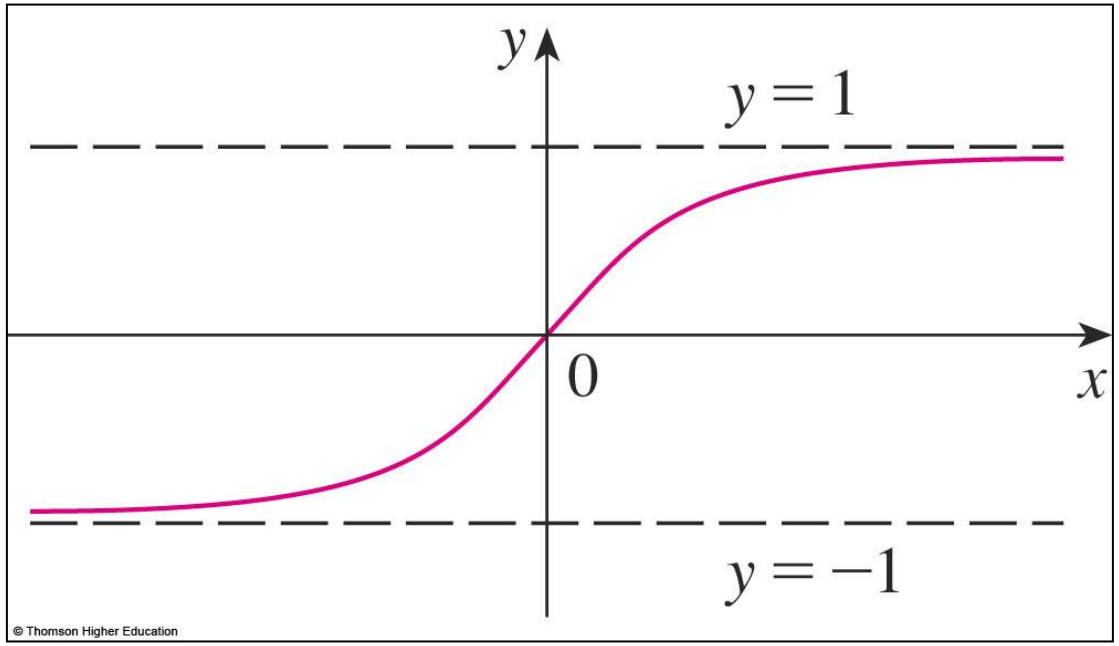




Note that sinh has domain $\mathbb{R}$ and range $\mathbb{R}$, whereas cosh has domain $\mathbb{R}$ and range $[1, \infty)$

The graph of tanh is shown.

It has the horizontal asymptotes $y = \pm 1$.



The hyperbolic functions satisfy a number of identities that are similar to well-known trigonometric identities.

Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

$$\sinh(-x) = -\sinh x$$

$$\cosh(-x) = \cosh x$$

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$\sinh(x + y) = \sinh x \cosh y + \cosh x \sinh y$$

$$\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$$

Example 1

- Prove:

a. $\cosh^2 x - \sinh^2 x = 1$

b. $1 - \tanh^2 x = \operatorname{sech}^2 x$

solution a

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4}$$

$$= \frac{4}{4} = 1$$

solution b

We start with the identity proved in (a): $\cosh^2 x - \sinh^2 x = 1$

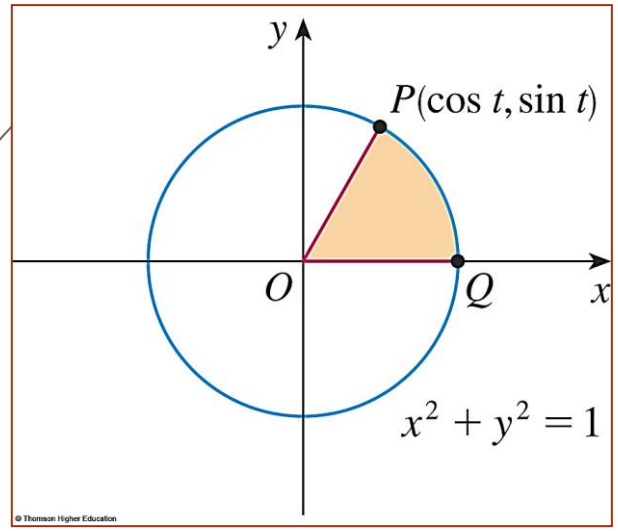
If we divide both sides by $\cosh^2 x$, we get:

$$1 - \frac{\sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x}$$

$$\text{or } 1 - \tanh^2 x = \operatorname{sech}^2 x$$

The identity proved in Example 1 a gives a clue to the reason for the name ‘hyperbolic’ functions, as follows.

If t is any real number, then the point P(cos t, sin t) lies on the unit circle
$x^2 + y^2 = 1$
because $\cos^2 t + \sin^2 t = 1$.



In fact, *t* can be interpreted as the radian measure of in the figure.

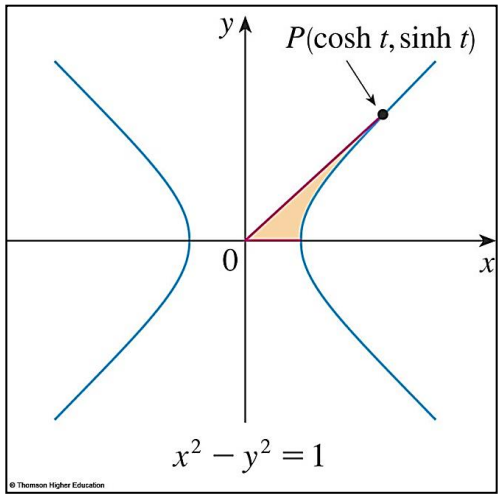
For this reason, the trigonometric functions are sometimes called circular functions.

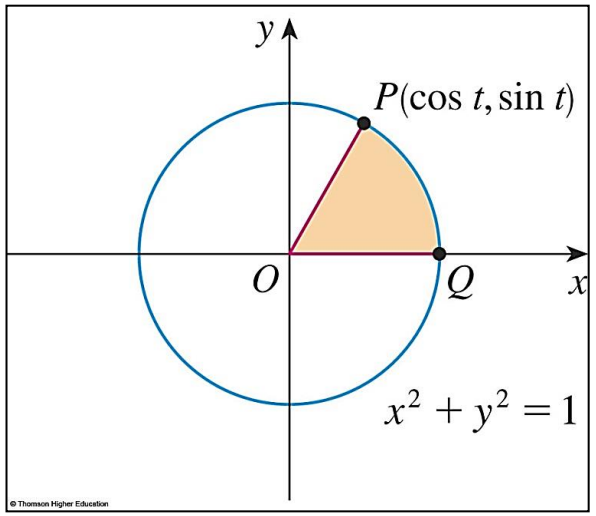
Likewise, if t is any real number, then the point P(cosh t, sinh t) lies on the right branch of the hyperbola $x^2 - y^2 = 1$ because $\cosh^2 t - \sin^2 t = 1$ and $\cosh t \geq 1$.

This time, t does not represent the measure of an angle.

However, it turns out that t represents twice the area of the shaded hyperbolic sector in the first figure.

This is just as in the trigonometric case *t* represents twice the area of the shaded circular sector in the second figure.





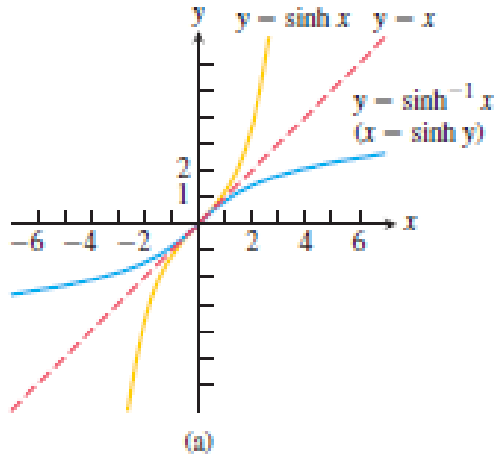
Inverse Hyperbolic functions

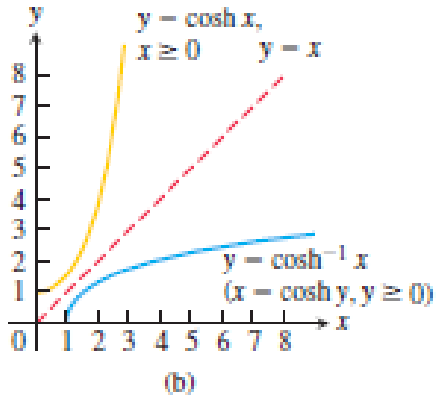
The inverse of the six hyperbolic functions are very useful in integration. We denote its inverse

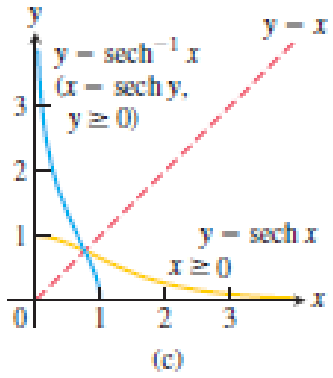
By $y = sinh^{-1}x \qquad -\infty < x < \infty$

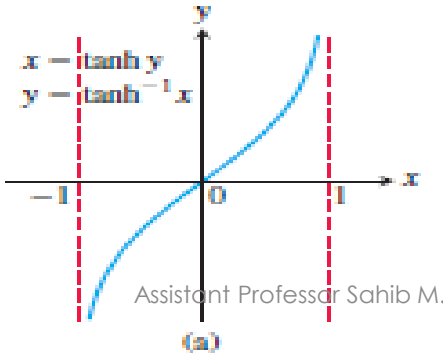
The restricted function $y = cosh^{-1}y \quad , \quad x \geq 1$ **and the function**

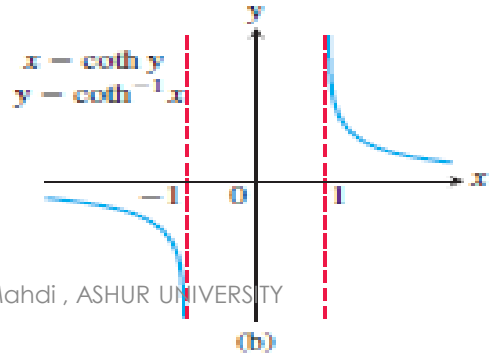
$$y = \operatorname{sech} x = \frac{1}{coshx} \quad an\ inverse\ .denoted\ by \quad y = sech^{-1}x$$

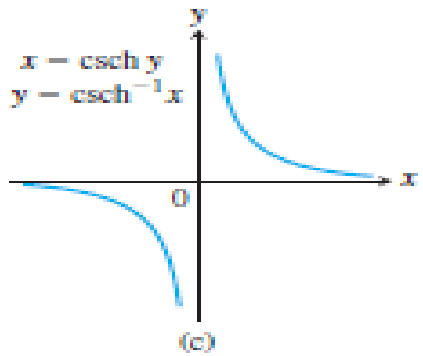












The hyperbolic tangent, cotangent and cosecant are have inverse

$y = tanh^{-1}x$,

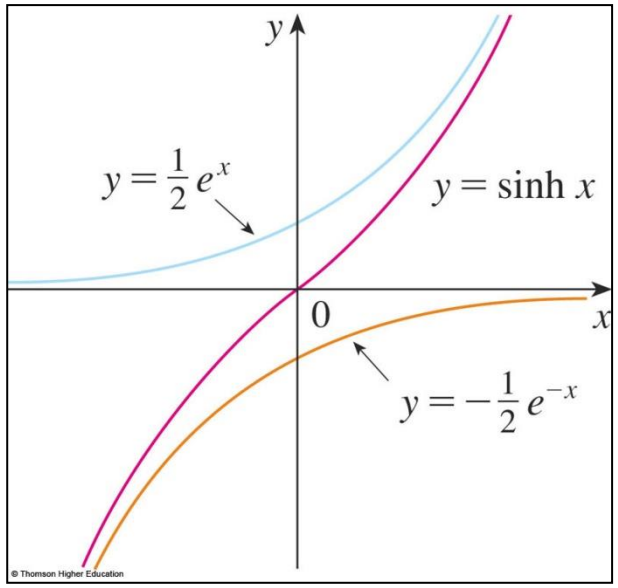
$y = coth^{-1}x$,

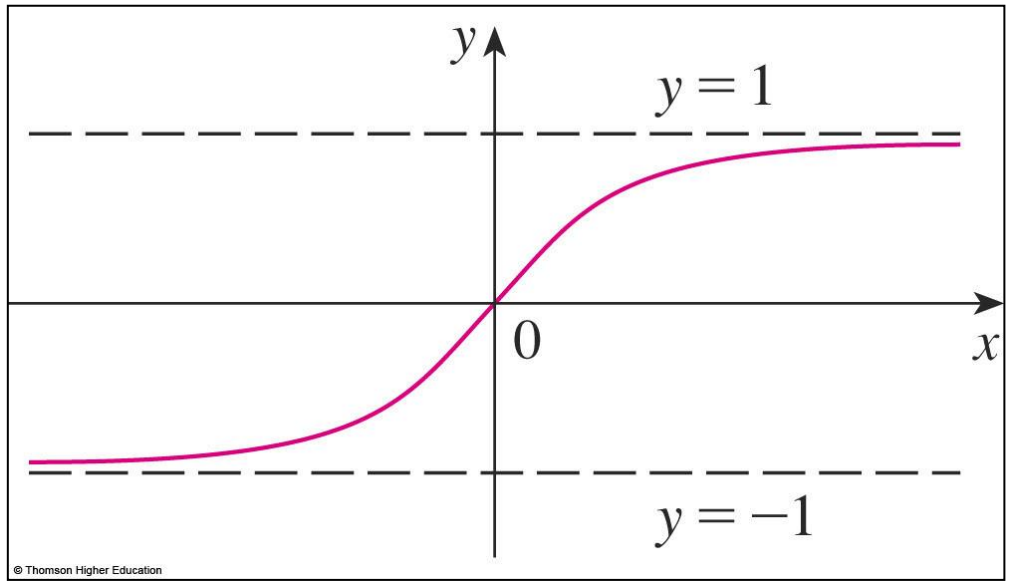
$y = csch^{-1}x$

INVERSE HYPERBOLIC FUNCTIONS

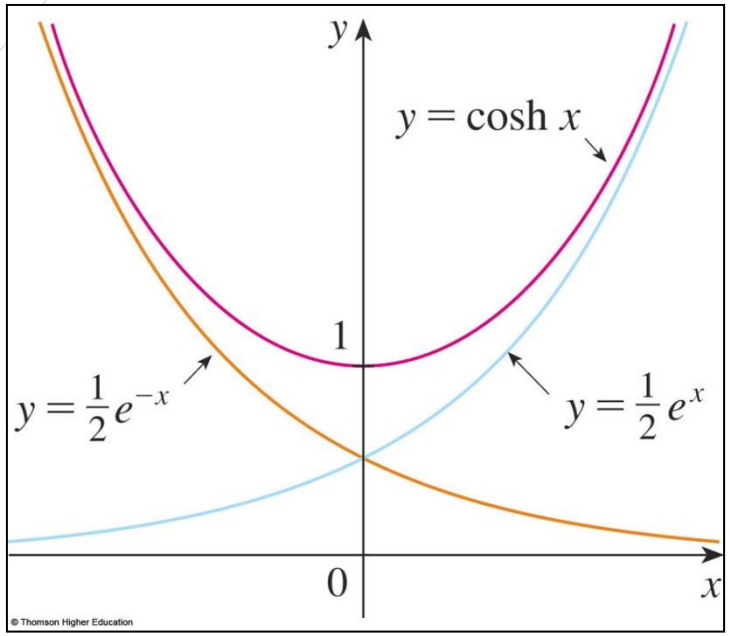
You can see from the figures that sinh and tanh are one-to-one functions.

So, they have inverse functions denoted by $\sinh^{-1}$ and $\tanh^{-1}$.





This figure shows that cosh is not one-to-one.
However, when restricted to the domain [0, ∞], it becomes one-to-one.



The inverse hyperbolic cosine function is defined as the inverse of this restricted function.

INVERSE FUNCTIONS **Definition 2**

$$y = \sinh^{-1} x \iff \sinh y = x$$

$$y = \cosh^{-1} x \iff \cosh y = x \text{ and } y \geq 0$$

$$y = \tanh^{-1} x \iff \tanh y = x$$

The remaining inverse hyperbolic functions are defined similarly.

Identities for inverse hyperbolic functions

$$\operatorname{sech}^{-1} x = \cosh^{-1} \frac{1}{x}$$

$$\operatorname{csch}^{-1} x = \sinh^{-1} \frac{1}{x}$$

$$\coth^{-1} x = \tanh^{-1} \frac{1}{x}$$

To prove that $sech^{-1}x = cosh^{-1}\frac{1}{x}$

$$Let\ \theta = cosh^{-1}\frac{1}{x} \quad \rightarrow \cosh\theta = \frac{1}{x}$$

$$x = \operatorname{sech}\theta \quad \rightarrow \theta = sech^{-1}x$$

$$\therefore \quad sech^{-1}x = cosh^{-1}\frac{1}{x}$$

Example: *Prove that* $sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$

$$Let\ y = sinh^{-1}x \quad \rightarrow \quad \sinh y = x$$

$$\frac{e^y - e^{-y}}{2} = x$$

$$e^y - e^{-y} = 2x \quad \rightarrow \quad e^y - \frac{1}{e^y} = 2x$$

$$e^{2y} - 1 = 2xe^y$$

$$e^{2y} - 2xe^y - 1 = 0$$

$$\therefore \quad e^y = \frac{2x \mp \sqrt{4x^2+4}}{2} = x \mp \sqrt{x^2+1}$$

$$e^y > 0 \quad neglect\ minus\ sign$$

$$e^y = x + \sqrt{x^2+1}$$

$$y = \ln\left(x+\sqrt{x^2+1}\right)$$

$$\therefore \quad sinh^{-1}x = \ln\left(x+\sqrt{x^2+1}\right)$$

Example: *Find* $\lim_{x\to\infty}(cosh^{-1}x - \ln x)$

$$y = cosh^{-1}x \quad \rightarrow \quad x = \cosh y = \frac{e^y + e^{-y}}{2}$$

$$e^y + e^{-y} = 2x$$

$$e^{2y} + 1 = 2e^y x$$

$$e^{2y} - 2xe^y + 1 = 0$$

$$e^y = \frac{2x \mp \sqrt{4x^2-4}}{2} = x \mp \sqrt{x^2-1}$$

$$e^y = x + \sqrt{x^2-1}$$

$$y = \ln\left(x+\sqrt{x^2-1}\right)$$

$$cosh^{-1}x = ln\left(x+\sqrt{x^2-1}\right)$$

$$= \lim_{x\to\infty}\left(\ln\left(x+\sqrt{x^2-1}\right) - \ln x\right)$$

$$= \lim_{x\to\infty}\left(\ln\frac{x+\sqrt{x^2-1}}{x}\right)$$

$$= \lim_{x\to\infty}\left(\ln\frac{1+\sqrt{1-\frac{1}{x^2}}}{1}\right) = \ln 2$$

Home work 5

Gives a value of sinh x or cosh x for below exercises.

Use the definitions and the identity

$\cosh^2 x - \sinh^2 x = 1$ to find the values of the remaining five hyperbolic functions.

1 - $\sinh x = -\frac{3}{4}$ **2 -** $\cosh x = \frac{17}{15}$, $x > 0$

Rewrite the expressions in Exercises 3–5 in terms of exponentials and simplify the results as much as you can.

3 - $2 \cosh (\ln x)$ **4 -** $\cosh 3x - \sinh 3x$

5 - $\ln (\cosh x + \sinh x) + \ln (\cosh x - \sinh x)$

6 - Prove the identitie

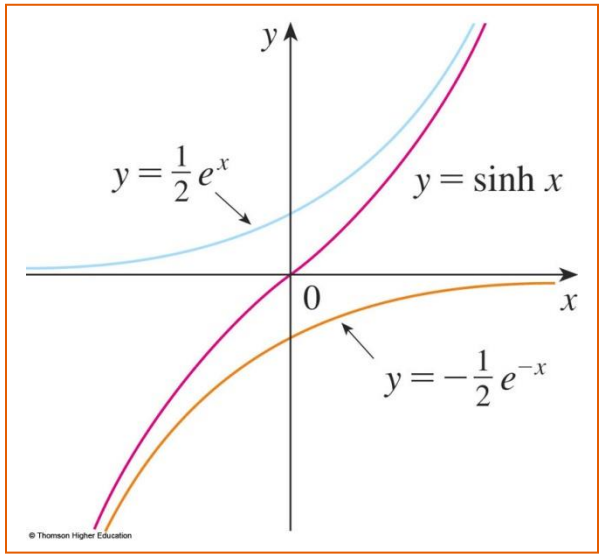
$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y,$$

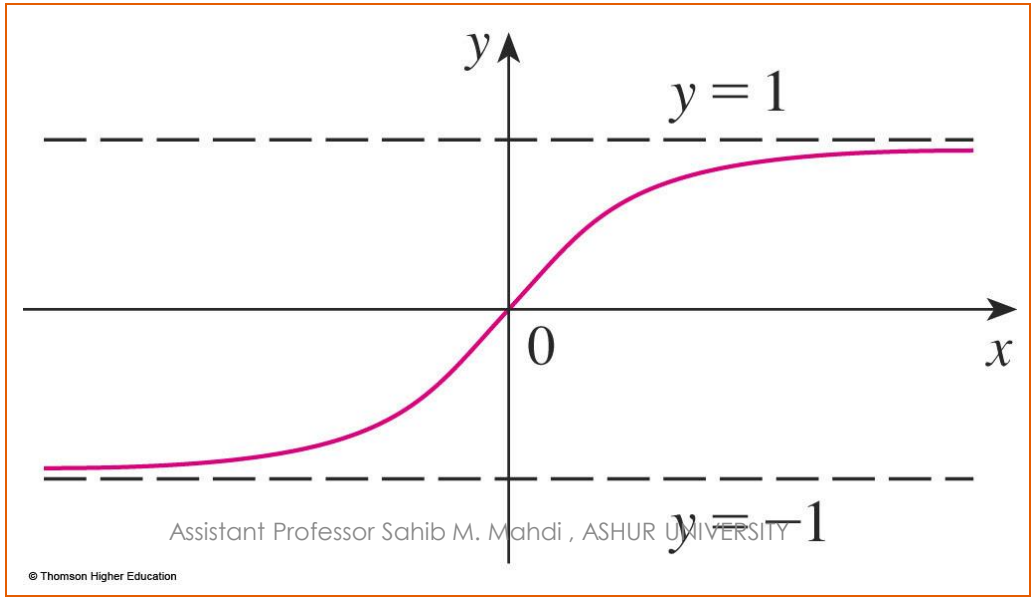
Then use it to show that

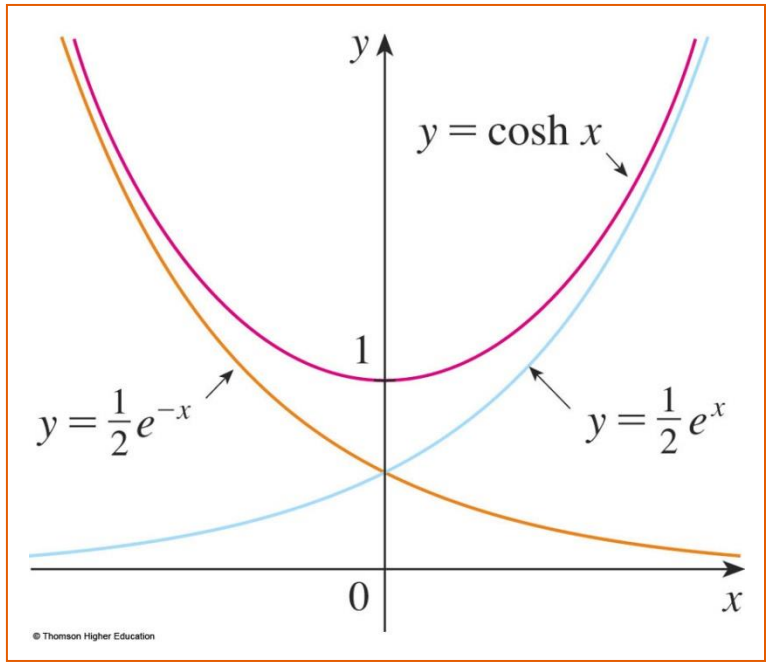
$$\sinh 2x = 2 \sinh x \cosh x.$$

INVERSE FUNCTIONS

By using these figures, we can sketch the graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$.

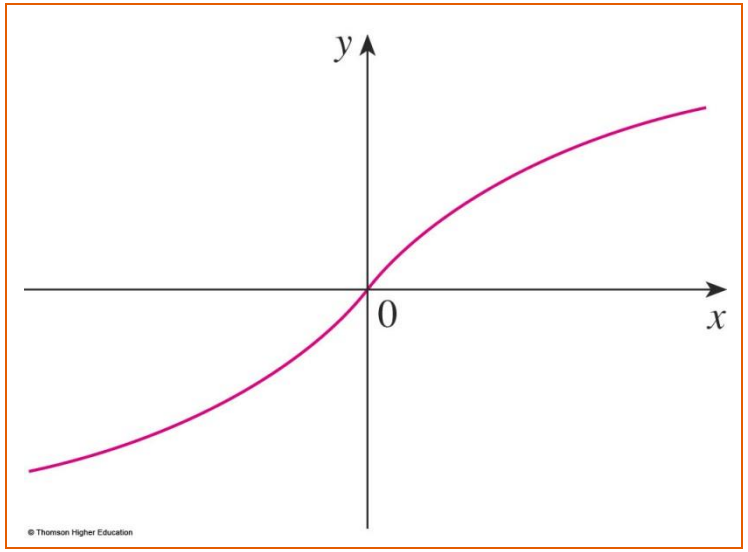


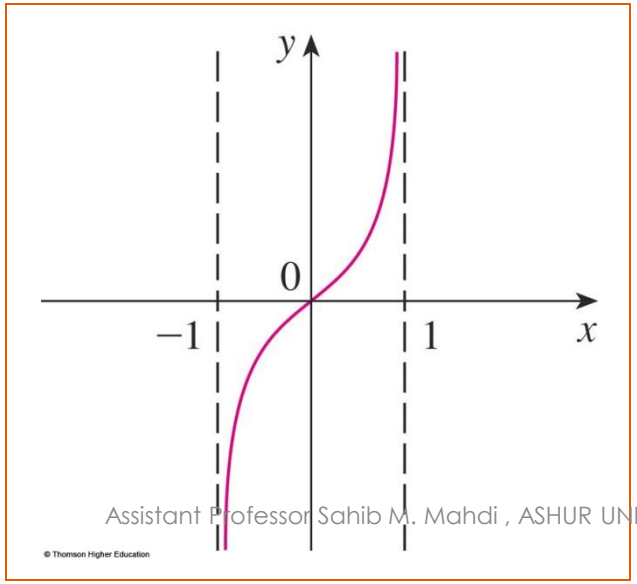


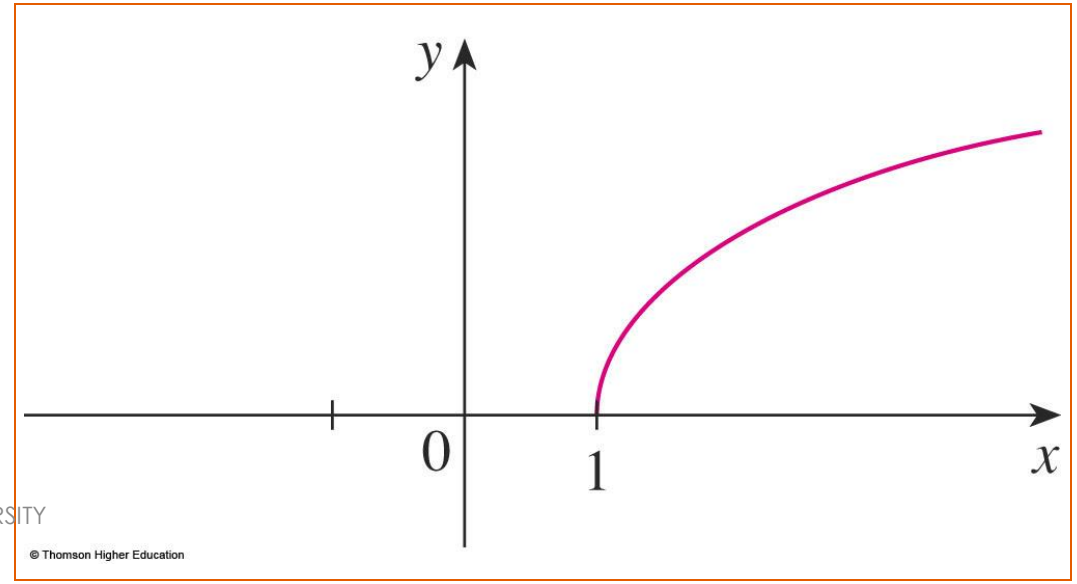


INVERSE FUNCTIONS

The graphs of $\sinh^{-1}$, $\cosh^{-1}$, and $\tanh^{-1}$ are displayed.







INVERSE FUNCTIONS

Since the hyperbolic functions are defined in terms of exponential functions, it's not surprising to learn that the inverse hyperbolic functions can be expressed in terms of logarithms.

In particular, we have:

$$\sinh^{-1} x = \ln\left(x + \sqrt{x^2 + 1}\right) \qquad x \in \mathbb{R}$$

$$\cosh^{-1} x = \ln\left(x + \sqrt{x^2 - 1}\right) \qquad x \geq 1$$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) \qquad -1 < x < 1$$

INVERSE FUNCTIONS

Example 3

Show that $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

Let $y = \sinh^{-1} x$. **Then,** $x = \sinh y = \dfrac{e^y - e^{-y}}{2}$

So, $e^y - 2x - e^{-y} = 0$

Or, multiplying by e^y, $e^{2y} - 2xe^y - 1 = 0$

This is really a quadratic equation in e^y:
$(e^y)^2 - 2x(e^y) - 1 = 0$

Solving by the quadratic formula,

we get: $e^y = \dfrac{2x \pm \sqrt{4x^2+4}}{2} = x \pm \sqrt{x^2+1}$

Note that $e^y > 0$,

but $x - \sqrt{x^2+1} < 0$

(because $x < \sqrt{x^2+1}$**).**

So, the minus sign is inadmissible and we have:

$$e^y = x + \sqrt{x^2+1}$$

Thus,

$$y = \ln(e^y) = \ln\left(x + \sqrt{x^2+1}\right)$$

Home work 6

Inverse Hyperbolic Functions

When hyperbolic function keys are not available on a calculator, it is still possible to evaluate the inverse hyperbolic functions by expressing them as logarithms, as shown here.

$$\sinh^{-1}x = \ln\left(x + \sqrt{x^2+1}\right), \qquad -\infty < x < \infty$$

$$\cosh^{-1}x = \ln\left(x + \sqrt{x^2-1}\right), \qquad x \geq 1$$

$$\tanh^{-1}x = \frac{1}{2}\ln\frac{1+x}{1-x}, \qquad |x| < 1$$

$$\operatorname{sech}^{-1}x = \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), \qquad 0 < x \leq 1$$

$$\operatorname{csch}^{-1}x = \ln\left(\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right), \qquad x \neq 0$$

$$\coth^{-1}x = \frac{1}{2}\ln\frac{x+1}{x-1}, \qquad |x| > 1$$

Use the formulas in the box here to express the numbers in Exercises 1– 6 in terms of natural logarithms.

1 - $\sinh^{-1}(-5/12)$

2 - $\cosh^{-1}(5/3)$

3 - $\tanh^{-1}(-1/2)$

4 - $\coth^{-1}(5/4)$

5 - $\operatorname{sech}^{-1}(3/5)$

6 – $\operatorname{csch}^{-1}\left(-1/\sqrt{3}\right)$