Differential Mathematics

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Lecture 2

Transcendental functions- trigonometric functions, and their inverses.

1.3 Trigonometric Functions

Angles

Angles are measured in degrees or radians. The number of **radians** in the central angle ***A′CB′*** within a circle of radius ***r*** is defined as the number of "radius units" contained in the arc ***s*** subtended by that central angle. If we denote this central angle by θ when measured in radians, this means that **θ = *s*/*r*** (Figure 1.36),

$$s = r\,\theta \quad (\theta \text{ in radians}) \ldots\ldots\ldots (1)$$

If the circle is a unit circle having radius ***r = 1***, then from Figure 1.36 and Equation (1), we see that the central angle θ measured in radians is just the length of the arc that the angle cuts from the unit circle. Since one complete revolution of the unit circle is ***360° or 2π*** radians, we have

$$\pi \text{ radians} = 180^\circ \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

and

1 radian = 180/π (≈57.3) degrees or **1 degree = π /180 (≈0.017)** radians.

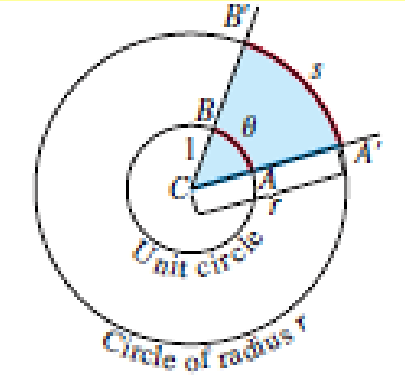


FIGURE 1.36 The radian measure of the central angle *A′CB′* is the number **θ = *s*/*r***. For a unit circle of radius ***r = 1***x**θ** is the length of arc *A'B'* that central angle *A'CB'* cuts from the unit circle.

TABLE 1.1 Angles measured in degrees and radians

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π

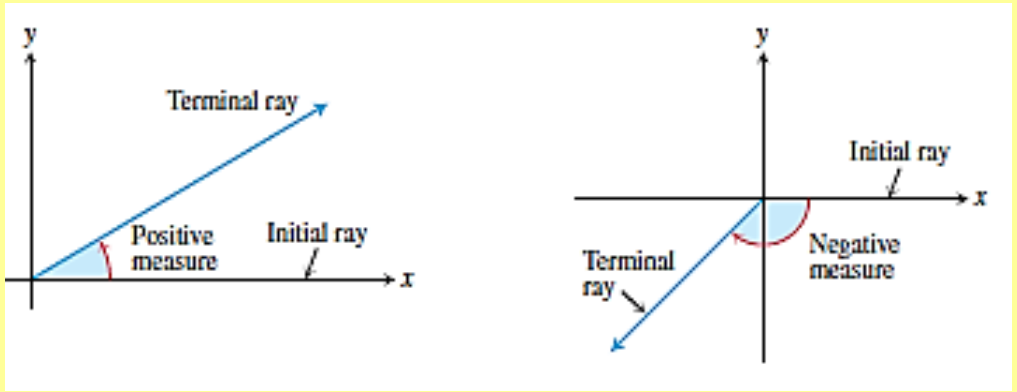


FIGURE 1.37 Angles in standard position in the *xy*-plane.

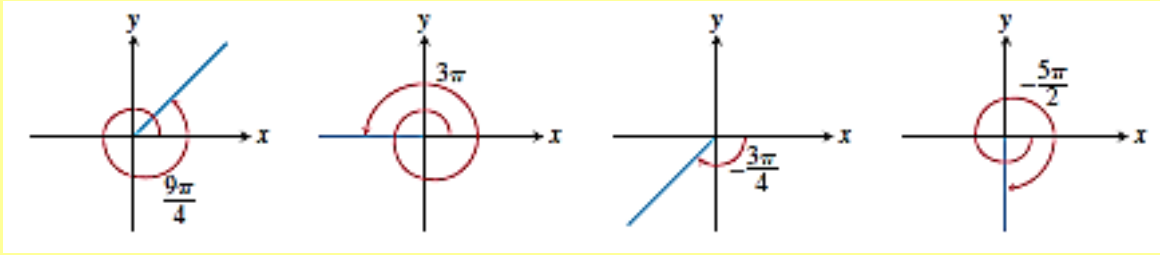


FIGURE 1.38 Nonzero radian measures can be positive or negative and can go beyond 2π.

The Six Basic Trigonometric Functions

sine:	$\sin\theta = \frac{y}{r}$	**cosecant:**	$\csc\theta = \frac{r}{y}$
cosine:	$\cos\theta = \frac{x}{r}$	**secant:**	$\sec\theta = \frac{r}{x}$
tangent:	$\tan\theta = \frac{y}{x}$	**cotangent:**	$\cot\theta = \frac{x}{y}$

These extended definitions agree with the right-triangle definitions when the angle is acute. Notice also that whenever the quotients are defined

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

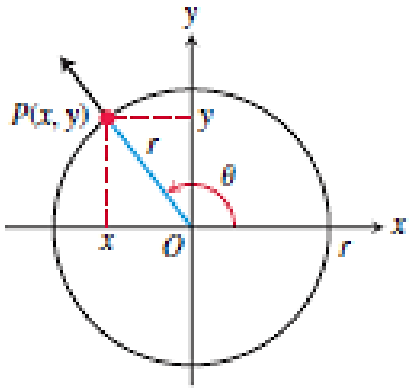
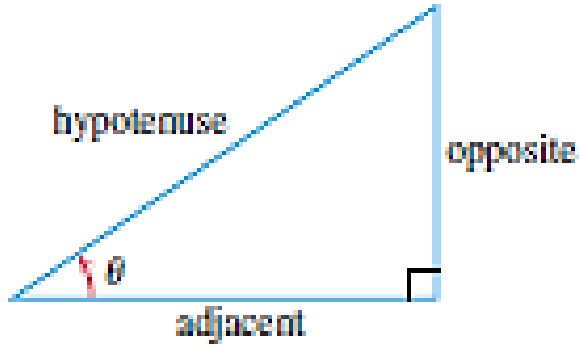


FIGURE 1.40 The trigonometric functions of a general angle θ are defined in terms of *x*, *y*, and *r*.



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 1.39 Trigonometric ratios of an acute angle.

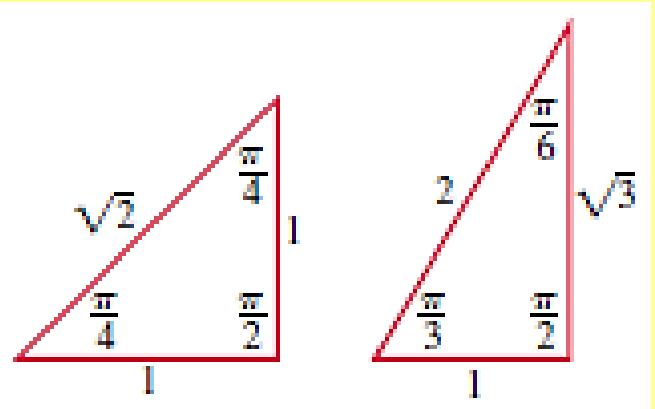


FIGURE 1.41 Radian angles and side lengths of two common triangles.

$$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \sin\frac{\pi}{6} = \frac{1}{2} \qquad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

$$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}} \qquad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2} \qquad \cos\frac{\pi}{3} = \frac{1}{2}$$

$$\tan\frac{\pi}{4} = 1 \qquad \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}} \qquad \tan\frac{\pi}{3} = \sqrt{3}$$

$$\sin\frac{2\pi}{3} = \frac{\sqrt{3}}{2}, \qquad \cos\frac{2\pi}{3} = -\frac{1}{2}, \qquad \tan\frac{2\pi}{3} = -\sqrt{3}.$$

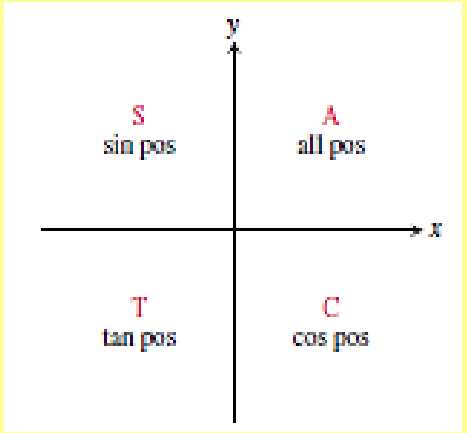


FIGURE 1.42 The CAST rule, remembered by the statement "Calculus Activates Student Thinking," tells which trigonometric functions are positive in each quadrant.

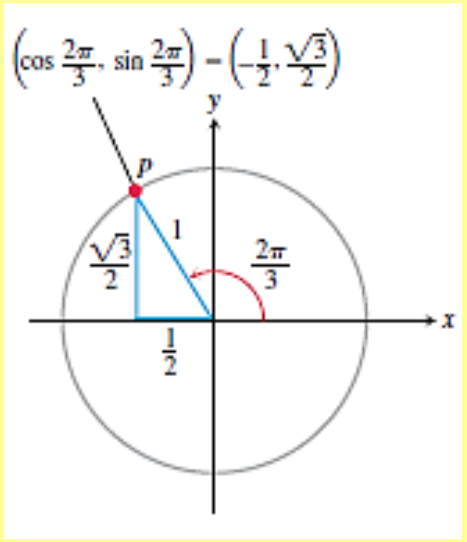


FIGURE 1.43 The triangle for calculating the sine and cosine of $2\pi/3$ radians. The side lengths come from the geometry of right triangles.

TABLE 1.2 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

Periodicity and Graphs of the Trigonometric Functions

When an angle of measure **θ** and an angle of measure **θ + 2π** are in standard position, their terminal rays coincide. The two angles therefore have the same trigonometric function values:
sin (θ + 2π) = sin θ, tan (θ + 2π) = tan θ, and so on. Similarly,
cos (θ - 2π) = cos θ, sin(θ - 2π) = sin θ, and so on. We describe this repeating behavior by saying that the six basic trigonometric functions are *periodic*.

DEFINITION A function $f(x)$ is **periodic** if there is a positive number p such that $f(x + p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.

Periods of Trigonometric Functions

Period π:
$$\tan(x + \pi) = \tan x$$
$$\cot(x + \pi) = \cot x$$

Period 2π:
$$\sin(x + 2\pi) = \sin x$$
$$\cos(x + 2\pi) = \cos x$$
$$\sec(x + 2\pi) = \sec x$$
$$\csc(x + 2\pi) = \csc x$$

When we graph trigonometric functions in the coordinate plane, we usually denote the independent variable by ***x*** instead of θ. Figure 1.44 shows that the tangent and cotangent functions have period ***p = π,*** and the other four functions have period **2π**. Also, the symmetries in these graphs reveal that the cosine and secant functions are even and the other four functions are odd (although this does not prove those results).

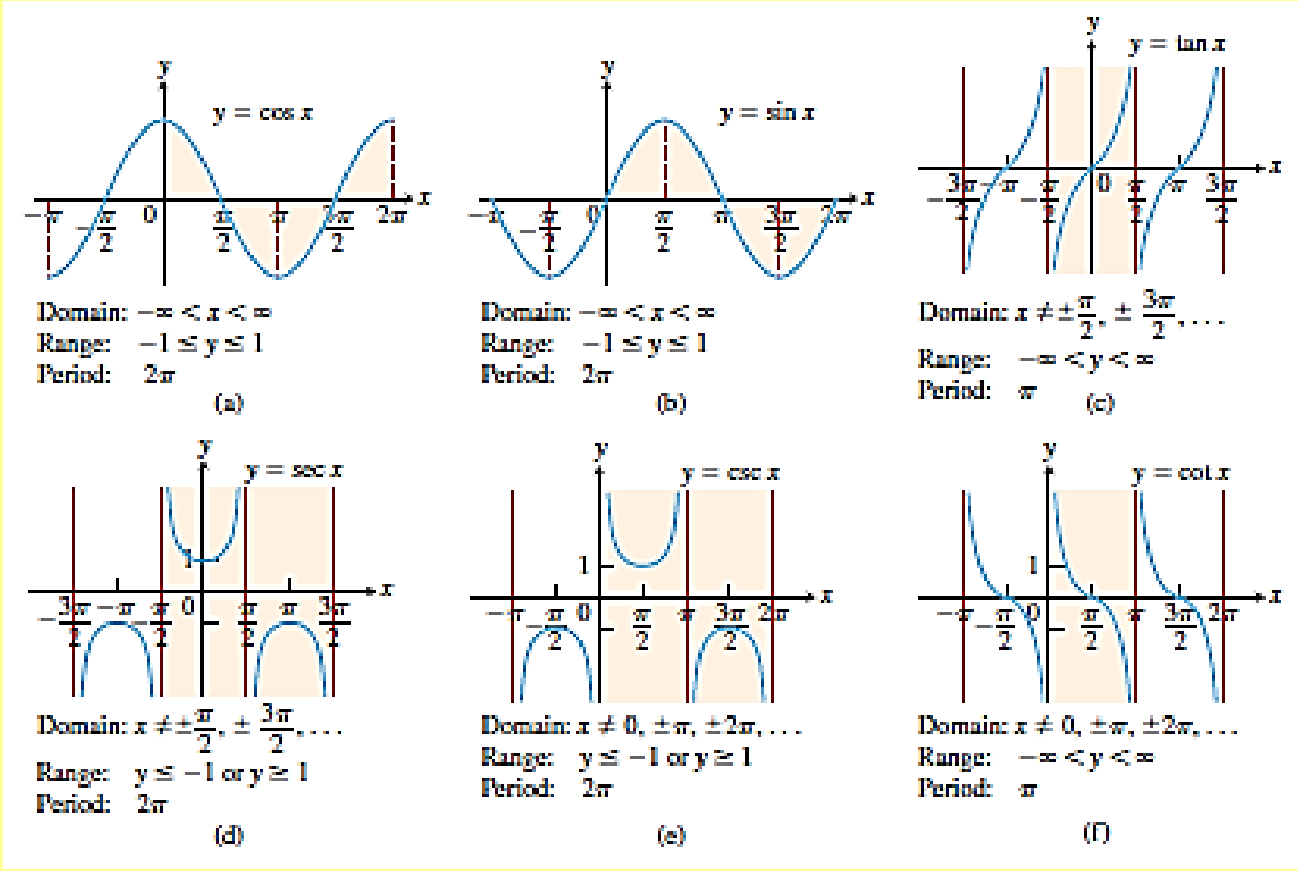


FIGURE 1.44 Graphs of the six basic trigonometric functions using radian measure. The shading for each trigonometric function indicates its periodicity.

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Trigonometric Identities

The coordinates of any point ***P(x, y)*** in the plane can be expressed in terms of the point's distance ***r*** from the origin and the angle θ that ray *OP* makes with the positive *x*-axis (Figure 1.40). Since ***x/r = cos θ*** and ***y/r = sin θ***, we have ***x = r cos θ, y = r sin θ.***

When ***r = 1*** we can apply the Pythagorean theorem to the reference right triangle in Figure 1.45 and obtain the equation

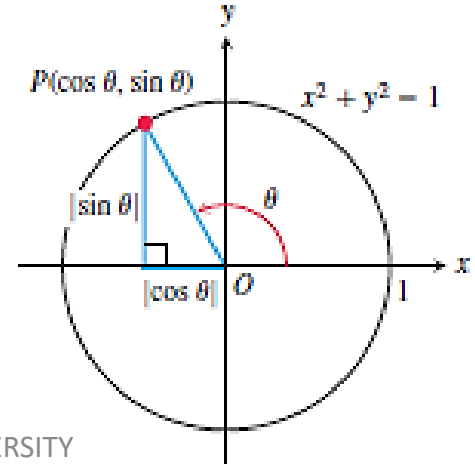


FIGURE 1.45 The reference triangle for a general angle θ.

$$\cos^2\theta + \sin^2\theta = 1 \qquad \text{..... (3)}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta$$
$$1 + \cot^2\theta = \csc^2\theta$$

The following formulas hold for all angles *A* and *B* (Exercise 58).

Addition Formulas

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\sin(A + B) = \sin A\cos B + \cos A\sin B \quad \text{.....(4)}$$

There are similar formulas for cos (*A* - *B*) and sin (*A* - *B*) (Exercises 35 and 36). All the trigonometric identities needed in this book derive from Equations (3) and (4). For example, substituting θ for both *A* and *B* in the addition formulas gives

Double-Angle Formulas

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$
$$\sin 2\theta = 2\sin\theta\cos\theta \quad \text{..... (5)}$$

Additional formulas come from combining the equations

$$\cos^2\theta + \sin^2\theta = 1, \qquad \cos^2\theta - \sin^2\theta = \cos 2\theta.$$

We add the two equations to get $2\cos^2\theta = 1 + \cos 2\theta$ and subtract the second from the first to get $2\sin^2\theta = 1 - \cos 2\theta$. This results in the following identities, which are useful in integral calculus.

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \qquad (6)$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \qquad (7)$$

The Law of Cosines

If *a*, *b*, and *c* are sides of a triangle *ABC* and if θ is the angle opposite *c*, then

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \quad \text{.....(8)}$$

This equation is called the **law of cosines**.

We can see why the law holds if we introduce coordinate axes with the origin at ***C*** and the positive *x*-axis along one side of the triangle, as in Figure 1.46. The coordinates of ***A*** are ***(b, 0);*** the coordinates of ***B*** are ***(a* cos θ, *a* sin θ).*** The square of the distance between ***A*** and ***B*** is therefore

$$c^2 = (a\cos\theta - b)^2 + (a\sin\theta)^2$$
$$= a^2(\cos^2\theta + \sin^2\theta) + b^2 - 2ab\cos\theta$$
$$= a^2 + b^2 - 2ab\cos\theta.$$

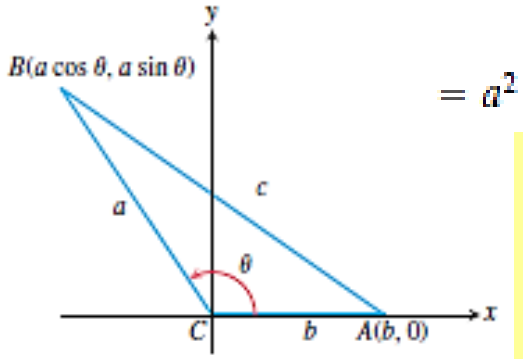


FIGURE 1.46 The square of the distance between A and B gives the law of cosines.

The law of cosines generalizes the Pythagorean theorem.If $\theta = \pi/2$, then $\cos\theta = 0$ and $c^2 = a^2 + b^2$.

Example 1 – *Evaluating Trigonometric Functions*

Evaluate the six trigonometric functions at each real number.

a. $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = \pi$

Solution:

For each *t*-value, begin by finding the corresponding point (x, y) on the unit circle. Then use the definitions of trigonometric functions.

Example1(a) – *Solution*

$t = \pi/6$ corresponds to the point $(x, y) = \left(\sqrt{3}/2, 1/2\right)$.

1 - $\sin\frac{\pi}{6} = y = \frac{1}{2}$

2 - $\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$

3 - $\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

4 - $\csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$

5 - $\sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

6 - $\cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$

Example1(b) – *Solution*

$t = 5\pi/4$ corresponds to the point $(x, y) = \left(-\sqrt{2}/2, -\sqrt{2}/2\right).$

1 - $\sin\dfrac{5\pi}{4} = y = -\dfrac{\sqrt{2}}{2}$

2 - $\cos\dfrac{5\pi}{4} = x = -\dfrac{\sqrt{2}}{2}$

3 - $\tan\dfrac{5\pi}{4} = \dfrac{y}{x} = \dfrac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$

4 - $\csc\dfrac{5\pi}{4} = \dfrac{1}{y} = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$

5 - $\sec\dfrac{5\pi}{4} = \dfrac{1}{x} = -\dfrac{2}{\sqrt{2}} = -\sqrt{2}$

6 - $\cot\dfrac{5\pi}{4} = \dfrac{x}{y} = \dfrac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$

Example1(c) – Solution

$t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

1 - $\sin\pi = y = 0$

2 - $\cos\pi = x = -1$

3 - $\tan\pi = \dfrac{y}{x} = \dfrac{0}{-1} = 0$

4 - $\csc\pi = \dfrac{1}{y}$ is undefined.

5 - $\sec\pi = \dfrac{1}{x} = \dfrac{1}{-1} = -1$

6 - $\cot\pi = \dfrac{x}{y}$ is undefined.

Example 2 –

Using the Period to Evaluate Sine and Cosine

Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$, **you have**

$$\sin\frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right)$$

$$= \sin\frac{\pi}{6}$$

$$= \frac{1}{2}.$$

$$\cos\frac{13\pi}{6} = \cos\left(2\pi + \frac{\pi}{6}\right)$$

$$= \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$$

Home work 3

Radians and Degrees

1 - On a circle of radius 10 m, how long is an arc that subtends a central angle of (**a**) $4\pi/5$ radians? (**b**) 110°?

2 - You want to make an 80° angle by marking an arc on the perimeter of a 12-in.-diameter disk and drawing lines from the ends of the arc to the disk's center. To the nearest tenth of an inch, how long should the arc be?

Evaluating Trigonometric Functions

3 - Copy and complete the following table of function values. If the function is undefined at a given angle, enter "UND." Do not use a calculator or tables.

θ	$-\pi$	$-2\pi/3$	0	$\pi/2$	$3\pi/4$
$\sin\theta$					
$\cos\theta$					
$\tan\theta$					
$\cot\theta$					
$\sec\theta$					
$\csc\theta$					

4 - one of sin x, cos x, and tan x is given. Find the other two if x lies in the specified interval.

A- $\sin x = \frac{3}{5}, \quad x \in \left[\frac{\pi}{2}, \pi\right]$

B- $\cos x = \frac{1}{3}, \quad x \in \left[-\frac{\pi}{2}, 0\right]$

C- $\tan x = \frac{1}{2}, \quad x \in \left[\pi, \frac{3\pi}{2}\right]$

Inverse Trigonometric Functions

A function is one-to-one if for every **x** there is exactly one y and for every **y** there is exactly one x.

But trigonometric functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one. The sine function increases from **-1** at **$x = -\pi/2$** to **+1** at **$x = \pi/2$**. By restricting its domain to the interval **$[-\pi/2, \pi/2]$** we make it one-to-one, so that it has an inverse **$\sin^{-1} x$** (Figure 1.64). Similar domain restrictions can be applied to all six trigonometric functions.

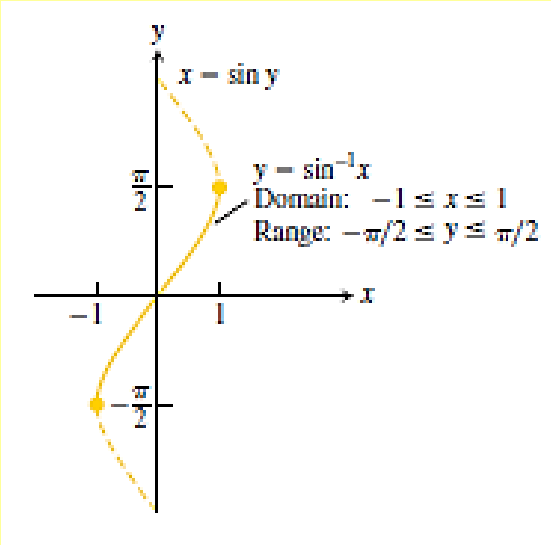
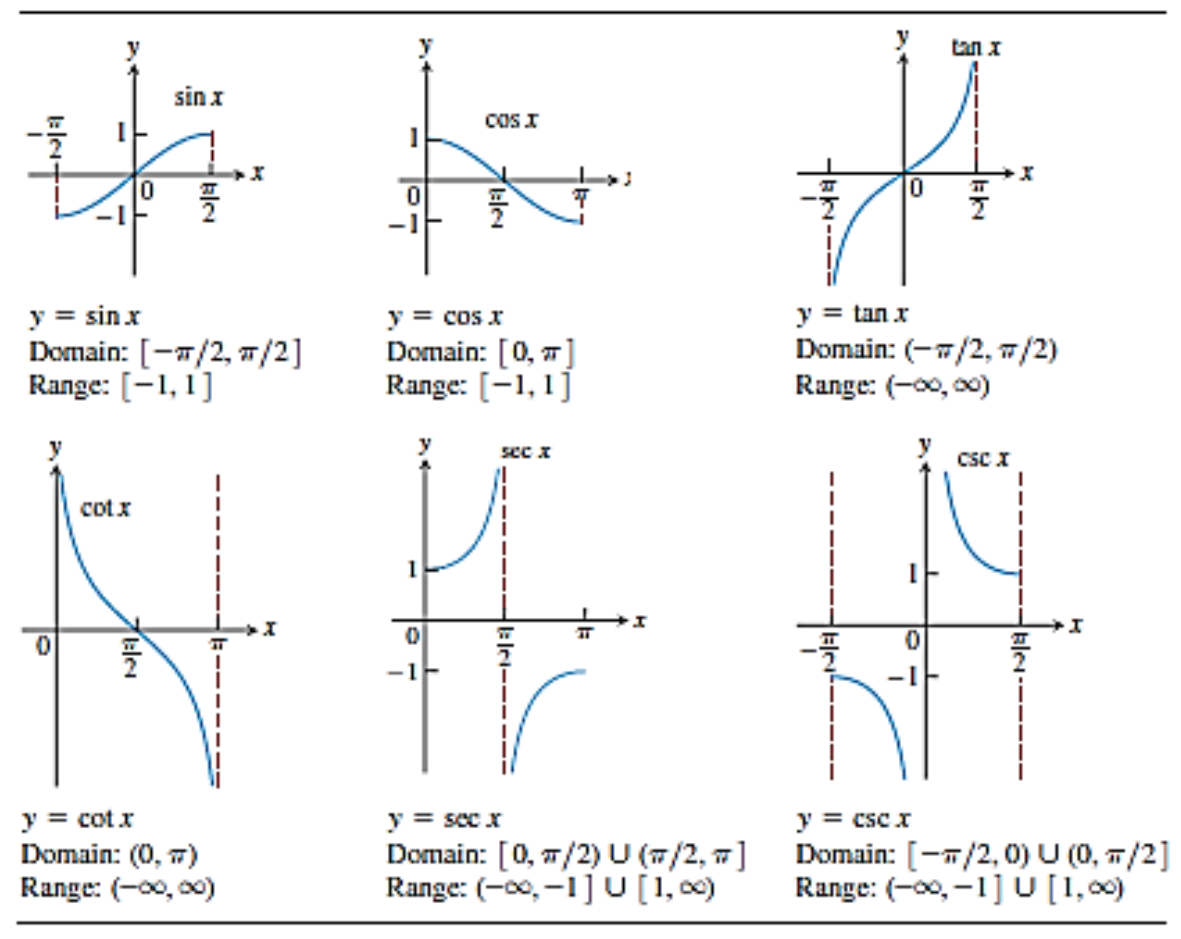


FIGURE 1.64 The graph of $y = \sin^{-1} x$.

Domain restrictions that make the trigonometric functions one-to-one



Since these restricted functions are now one-to-one, they have inverses, which we denote by

$y = \sin^{-1} x$	or	$y = \arcsin x$
$y = \cos^{-1} x$	or	$y = \arccos x$
$y = \tan^{-1} x$	or	$y = \arctan x$
$y = \cot^{-1} x$	or	$y = \operatorname{arccot} x$
$y = \sec^{-1} x$	or	$y = \operatorname{arcsec} x$
$y = \csc^{-1} x$	or	$y = \operatorname{arccsc} x$

Caution The -1 in the expressions for the inverse means "inverse." It does *not* mean reciprocal. For example, the *reciprocal* of **sin x is $(\sin x)^{-1} = 1/\sin x = \csc x$.**

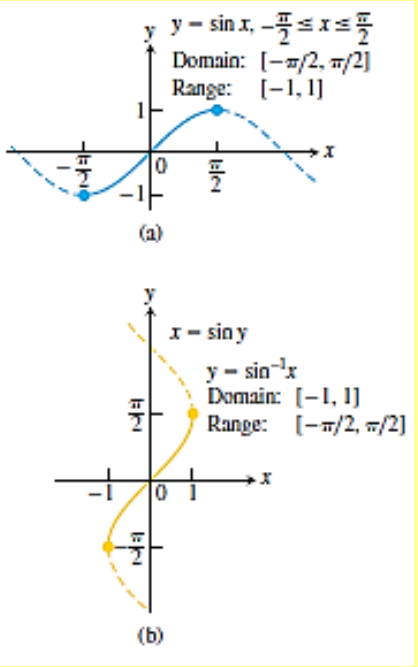


FIGURE 1.65 The graphs of (a) $y = \sin x$, $-\pi/2 \le x \le \pi/2$, and (b) its inverse, $y = \sin^{-1} x$. The graph of $\sin^{-1} x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \sin y$.

The Arcsine and Arccosine Functions

We define the arcsine and arccosine as functions whose values are angles (measured in radians) that belong to restricted domains of the sine and cosine functions.

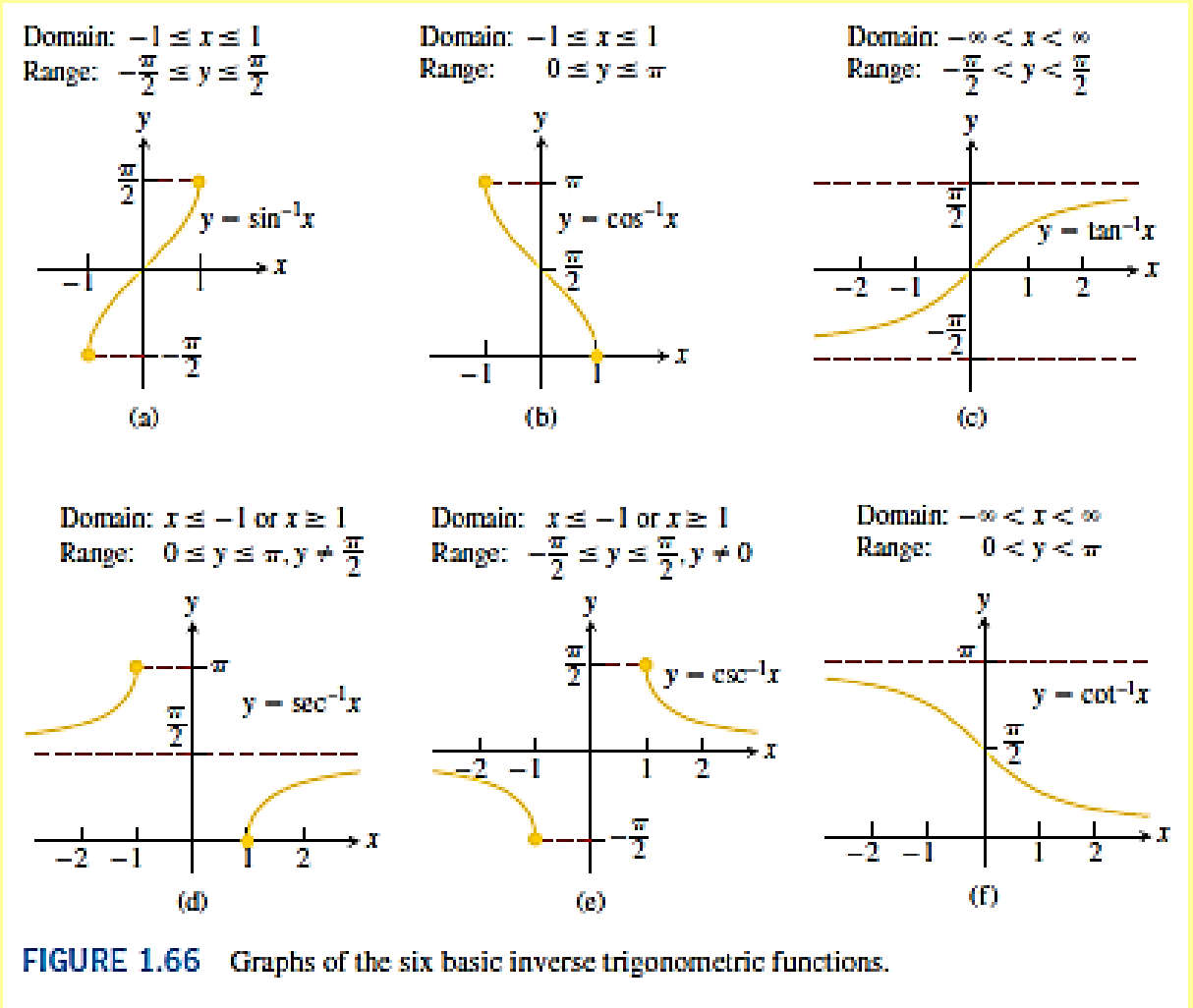


FIGURE 1.66 Graphs of the six basic inverse trigonometric functions.

EXAMPLE 8 Evaluate **(a)** $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ and **(b)** $\cos^{-1}\left(-\frac{1}{2}\right)$

Solution

(a) We see that $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$

because $\sin(\pi/3) = \sqrt{3}/2$ and $\pi/3$ belongs to the range $[-\pi/2, \pi/2]$ of the arcsine function. See Figure 1.68a.

(b) We have $\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}$

because $\cos(2\pi/3) = -1/2$ and $2\pi/3$ belongs to the range $[0, \pi]$ of the arccosine function. See Figure 1.68b. ■

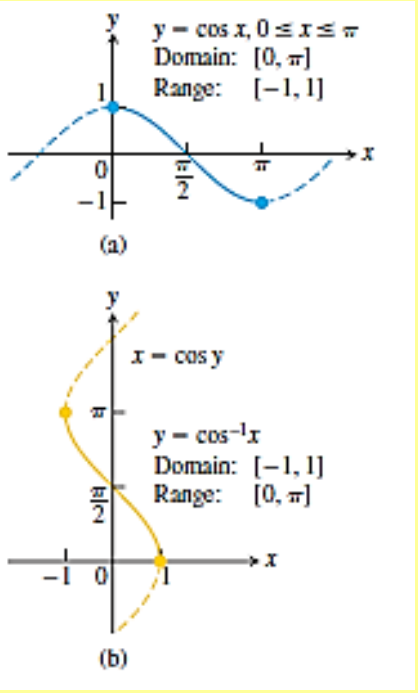


FIGURE 1.67 The graphs of (a) $y = \cos x$, $0 \le x \le \pi$, and (b) its inverse, $y = \cos^{-1}x$. The graph of $\cos^{-1}x$, obtained by reflection across the line $y = x$, is a portion of the curve $x = \cos y$.

Using the same procedure illustrated in Example 8, we can create the following table of common values for the arcsine and arccosine functions.

x	$\sin^{-1}x$	$\cos^{-1}x$
$\sqrt{3}/2$	$\pi/3$	$\pi/6$
$\sqrt{2}/2$	$\pi/4$	$\pi/4$
$1/2$	$\pi/6$	$\pi/3$
$-1/2$	$-\pi/6$	$2\pi/3$
$-\sqrt{2}/2$	$-\pi/4$	$3\pi/4$
$-\sqrt{3}/2$	$-\pi/3$	$5\pi/6$

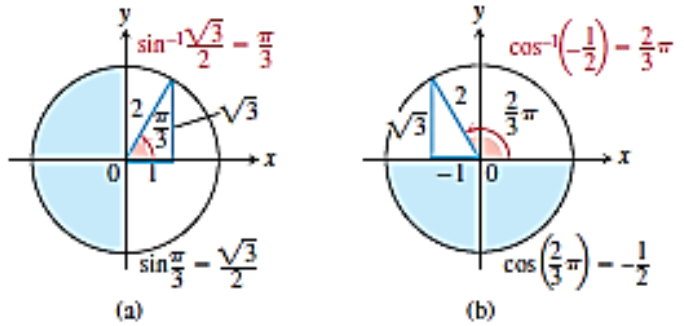


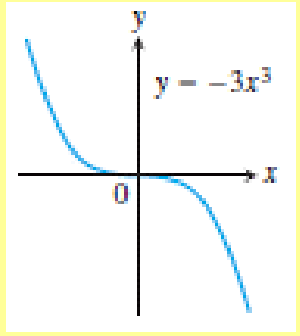
FIGURE 1.68 Values of the arcsine and arccosine functions (Example 8).

Home work 4

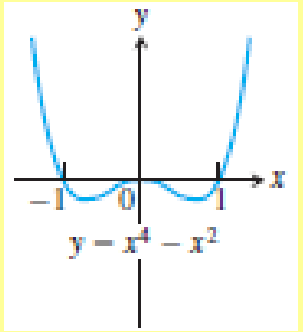
Identifying One-to-One Functions Graphically

Which of the functions graphed in Exercises 1–4 are one-to-one, and which are not?

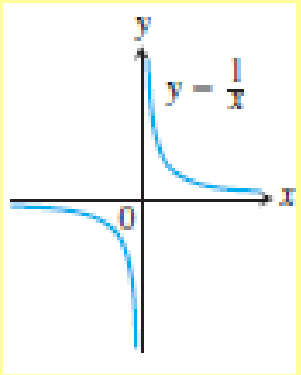
1 -



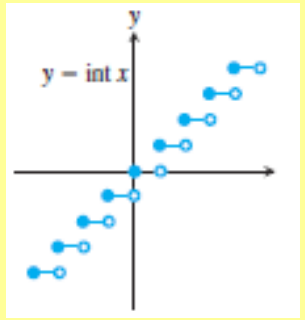
2 -



3 -



4 -



5 - determine from its graph if the function is one-to one.

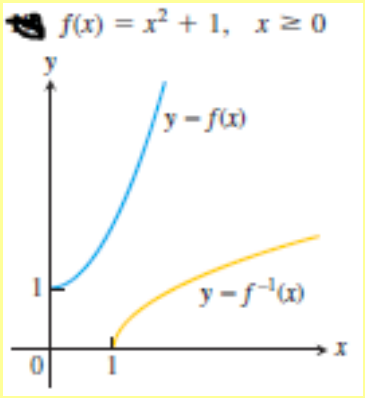
A - $$f(x) = \begin{cases} 3 - x, & x < 0 \\ 3, & x \geq 0 \end{cases}$$

B - $$f(x) = \begin{cases} 2x + 6, & x \leq -3 \\ x + 4, & x > -3 \end{cases}$$

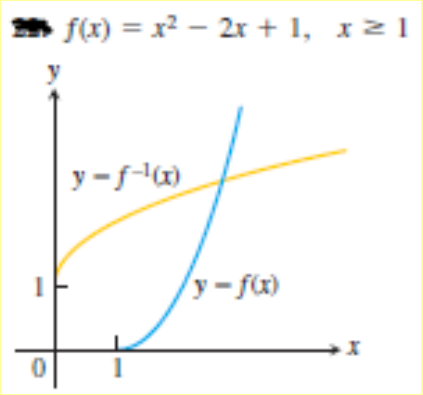
Formulas for Inverse Functions

Each of Exercises 6-8 gives a formula for a function y = f(x) and shows the graphs of f and f^{-1}. Find a formula for f^{-1} in each case.

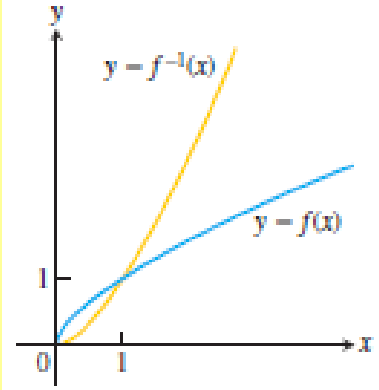
6 - $f(x) = x^2 + 1, \quad x \geq 0$



7 - $f(x) = x^2 - 2x + 1, \quad x \geq 1$



8 - $f(x) = x^{2/3}, \quad x \geq 0$



Inverses of Lines

a. Find the inverse of the function $f(x) = mx$, where m is a constant different from zero.

b. What can you conclude about the inverse of a function $y = f(x)$ whose graph is a line through the origin with a nonzero slope m?