Differential Mathematics

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Lecture 1

Limits and Continuity

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Concept of a Function

- A FUNCTION expresses the relationship of one variable or a group of variables (called the domain) with another variables(called the range) by associating every member in the domain to a unique member in range.

$$y = x^2$$

y is a function of x, and the relation $y = x^2$ describes a function. We notice that with such a relation, every value of x corresponds to one (and only one) value of y.
Since the value of y depends on a given value of x, we call y the dependent variable and x the independent variable and of the function $y = x^2$.

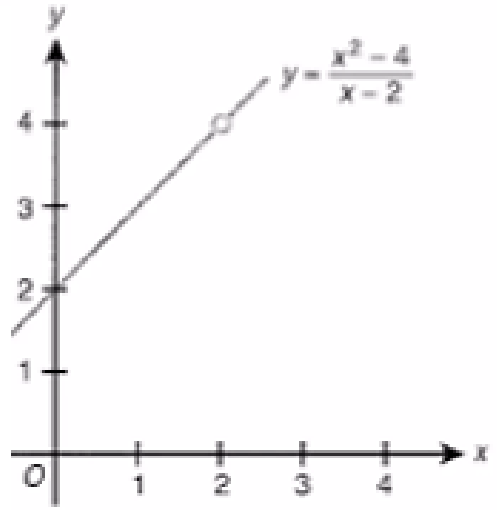
Limits and Continuity

We use limits to describe the way a function varies. Some functions vary *continuously*; small changes in **x** produce only small changes in $f(x)$. Other functions can have values that jump, vary erratically, or tend to increase or decrease without bound. The notion of limit gives a precise way to distinguish between these behaviors.

The Idea of Limits

Consider the function $f(x) = \dfrac{x^2 - 4}{x - 2}$

x	1.9	1.99	1.999	1.9999	**2**	2.0001	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	3.9999	**un-defined**	4.0001	4.001	4.01	4.1



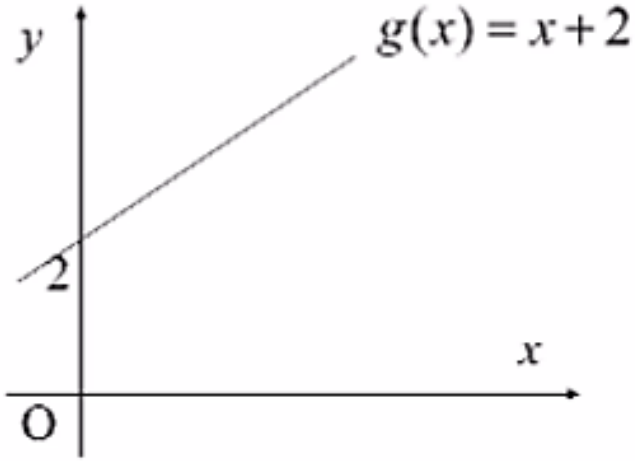
2.1 Rates of Change and Tangents to Curves

Calculus is a tool that helps us understand how a change in one quantity is related to a change in another. How does the speed of a falling object change as a function of time?

How does the level of water in a barrel change as a function of the amount of liquid poured into it?

Consider the function $g(x) = x + 2$

x	1.9	1.99	1.999	1.9999	**2**	2.0001	2.001	2.01	2.1
$g(x)$	3.9	3.99	3.999	3.9999	**4**	4.0001	4.001	4.01	4.1



If a function $f(x)$ is a continuous at x_0, then $\lim_{x \to x_0} f(x) = f(x_0)$.

approaches to, but not equal to

$$\lim_{x \to 2} f(x) = 4$$

$$\lim_{x \to 2} g(x) = 4$$

A function $f(x)$ has limit l at x_0 if $f(x)$ can be made as close to l as we please by taking x sufficiently close to (but not equal to) x_0. We write

$$\lim_{x \to x_0} f(x) = l$$

Average and Instantaneous Speed

In the late sixteenth century, Galileo discovered that a solid object dropped from rest (not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling. This type of motion is called **free fall**. It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object. If y denotes the distance fallen in feet after t seconds, then Galileo's law is

$$y = 16t^2,$$

where 16 is the (approximate) constant of proportionality. (If y is measured in meters, the constant is 4.9.)

A moving object's **average speed** during an interval of time is found by dividing the distance covered by the time elapsed. The unit of measure is length per unit time: kilometers per hour, feet (or meters) per second, or whatever is appropriate to the problem at hand.

Example 1 A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution The average speed of the rock during a given time interval is the change in distance, y, divided by the length of the time interval, t. (Increments like y and t are reviewed in Appendix 3, and pronounced "delta y" and "delta t.") Measuring distance in feet and time in seconds, we have the following calculations:

(a) For the first 2 sec: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(0)^2}{2 - 0} = 32\frac{\text{ft}}{\text{sec}}$$

(b) From sec 1 to sec 2: $$\frac{\Delta y}{\Delta t} = \frac{16(2)^2 - 16(1)^2}{2 - 1} = 48\frac{\text{ft}}{\text{sec}}$$

We want a way to determine the speed of a falling object at a single instant t_0, instead of using its average speed over an interval of time. To do this, we examine what happens when we calculate the average speed over shorter and shorter time intervals starting at t_0.

Example 2 Find the speed of the falling rock in Example 1 at $t = 1$ and $t = 2$ sec.

Solution We can calculate the average speed of the rock over a time interval $(t_0, t_0 + h)$, having length $\Delta t = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}.$$

TABLE 2.1 Average speeds over short time intervals $[t_0, t_0 + h]$

Average speed: $\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h}$

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

Average Rates of Change and Secant Lines

Given any function $y = f(x)$, we calculate the average rate of change of y with respect to x over the interval $[x_1, x_2]$ by dividing the change in the value of y, $\Delta y = f(x_2) - f(x_1)$, by the length $\Delta x = x_2 - x_1 = h$ of the interval over which the change occurs. (We use the symbol h for Δx to simplify the notation here and later on.)

DEFINITION The **average rate of change** of $y = f(x)$ with respect to x over the interval $[x_1, x_2]$ is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \qquad h \neq 0.$$

Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is a **secant** to the curve. Thus, the average rate of change of f from x_1 to x_2 is identical with the slope of secant PQ. Let's consider what happens as the point Q approaches the point P along the curve, so the length h of the interval over which the change occurs approaches zero. We will see that this procedure leads to defining the slope of a curve at a point.

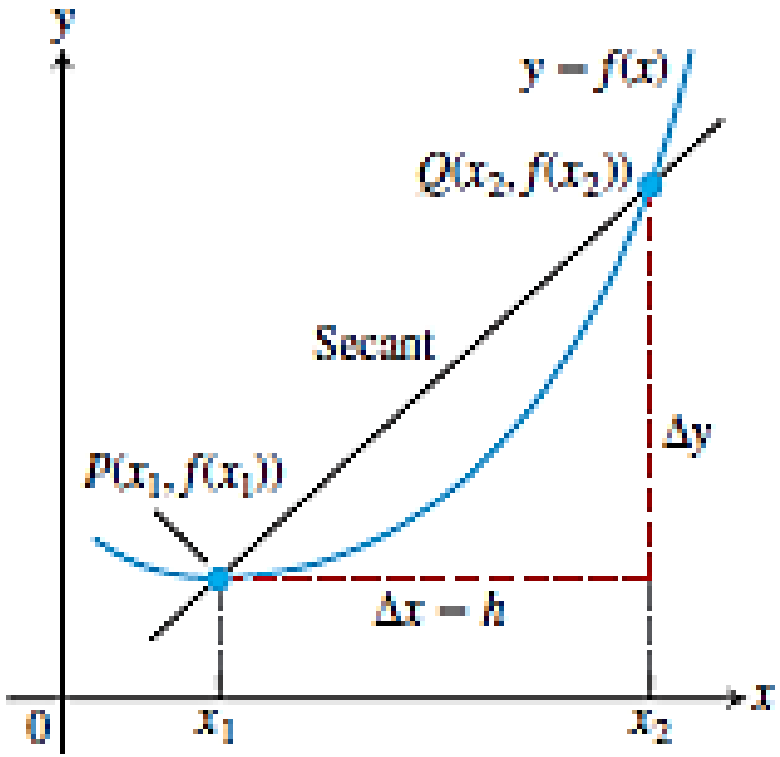
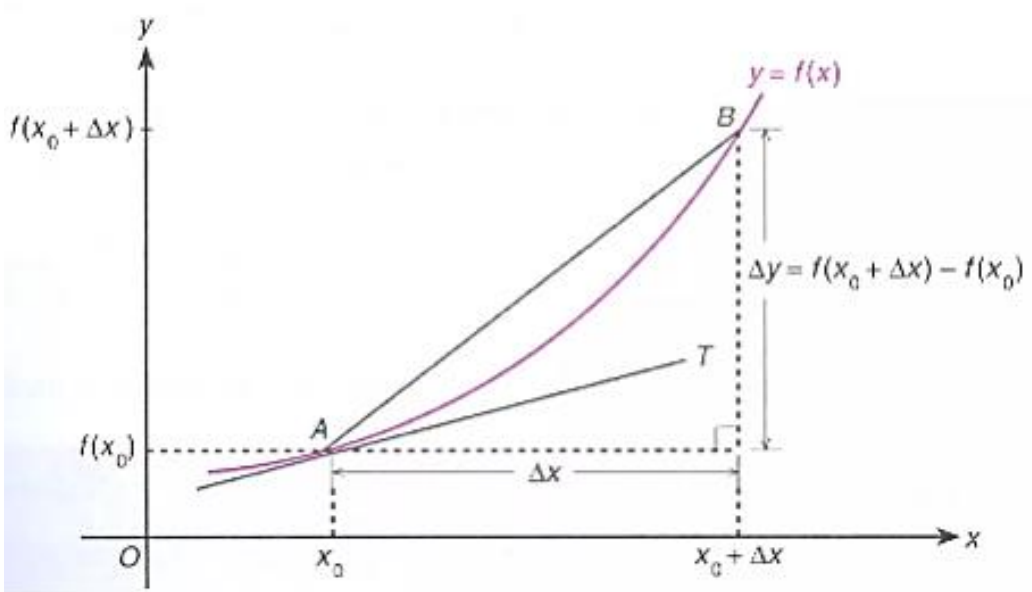


FIGURE 2.1 A secant to the graph $y = f(x)$. Its slope is $\Delta y / \Delta x$, the average rate of change of f over the interval $[x_1, x_2]$.

Defining the Slope of a Curve

We know what is meant by the slope of a straight line, which tells us the rate at which it rises or falls its rate of change as a linear function. But what is meant by the *slope of a curve* at a point **A** on the curve? If there is a *tangent* line to the curve at **A** a line that just touches the curve like the tangent to a circle it would be reasonable to identify *the slope of the tangent* as the slope of the curve at **A**. So we need a precise meaning for the tangent at a point on a curve.



The Slope of the Tangent to a Curve

The slope of the tangent to a curve $y = f(x)$ with respect to x is defined as

$$\text{Slope of } AT = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

provided that the limit exists.

EXAMPLE 3 Find the slope of the parabola $y = x^2$ at the point $P(2, 4)$. Write an equation for the tangent to the parabola at this point.

Solution We begin with a secant line through $P(2, 4)$ and $Q(2 + h, (2 + h)^2)$ nearby. We then write an expression for the slope of the secant PQ and investigate what happens to the slope as Q approaches P along the curve:

$$\text{Secant slope} = \frac{\Delta y}{\Delta x} = \frac{(2 + h)^2 - 2^2}{h} = \frac{h^2 + 4h + 4 - 4}{h}$$

$$= \frac{h^2 + 4h}{h} = h + 4.$$

If $h > 0$, then Q lies above and to the right of P, as in Figure 2.4. If $h < 0$, then Q lies to the left of P (not shown). In either case, as Q approaches P along the curve, h approaches zero and the secant slope $h + 4$ approaches 4. We take 4 to be the parabola's slope at P.

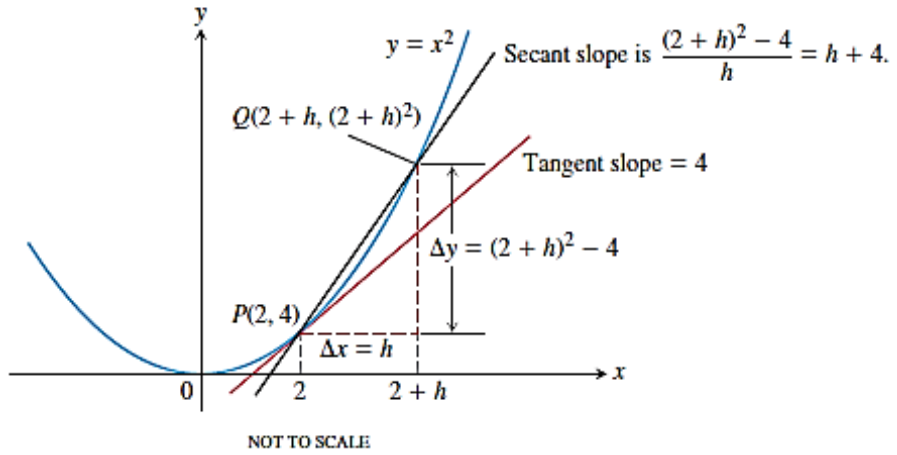


FIGURE 2.4 Finding the slope of the parabola $y = x^2$ at the point $P(2, 4)$ as the limit of secant slopes (Example 3).

The tangent to the parabola at P is the line through P with slope 4:

$y = 4 + 4(x - 2)$ **Point-slope equation**

$y = 4x - 4.$

Example 4 Figure 2.5 shows how a population p of fruit flies (*Drosophila*) grew in a 50-day experiment. The number of flies was counted at regular intervals, the counted values plotted with respect to time t, and the points joined by a smooth curve (colored blue in Figure 2.5). Find the average growth rate from day 23 to day 45.

Solution There were 150 flies on day 23 and 340 flies on day 45. Thus the number of flies increased by 340 - 150 = 190 in 45 - 23 = 22 days. The average rate of change of the population from day 23 to day 45 was

$$\text{Average rate of change: } \frac{\Delta p}{\Delta t} = \frac{340 - 150}{45 - 23} = \frac{190}{22} \approx 8.6 \text{ flies/day.}$$

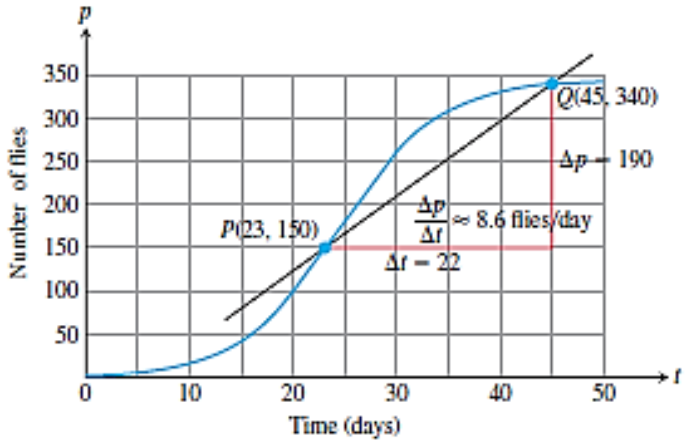


FIGURE 2.5 Growth of a fruit fly population in a controlled experiment. The average rate of change over 22 days is the slope $\Delta p/\Delta t$ of the secant line (Example 4).

This average is the slope of the secant through the points P and Q on the graph in Figure 2.5.

The average rate of change from day 23 to day 45 calculated in Example 4 does not tell us how fast the population was changing on day 23 itself. For that we need to examine time intervals closer to the day in question.

Example 5 How fast was the number of flies in the population of Example 4 growing on day 23?

Solution To answer this question, we examine the average rates of change over increasingly short time intervals starting at day 23. In geometric terms, we find these rates by calculating the slopes of secants from P to Q, for a sequence of points Q approaching P along the curve (Figure 2.6).

Q	Slope of $PQ = \Delta p/\Delta t$ (flies/day)
(45, 340)	$\frac{340-150}{45-23} \approx 8.6$
(40, 330)	$\frac{330-150}{40-23} \approx 10.6$
(35, 310)	$\frac{310-150}{35-23} \approx 13.3$
(30, 265)	$\frac{265-150}{30-23} \approx 16.4$

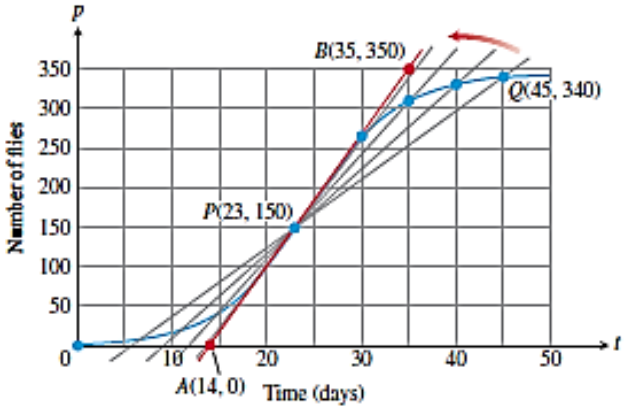


FIGURE 2.6 The positions and slopes of four secants through the point P on the fruit fly graph (Example 5).

The values in the table show that the secant slopes rise from 8.6 to 16.4 as the t-coordinate of Q decreases from 45 to 30, and we would expect the slopes to rise slightly higher as t continued on toward 23. Geometrically, the secants rotate counterclockwise about P and seem to approach the red tangent line in the figure. Since the line appears to pass through the points (14, 0) and (35, 350), it has slope

$$\frac{350-0}{35-14} = 16.7 \text{ flies/day (approximately).}$$

On day 23 the population was increasing at a rate of about 16.7 flies / day.

Home Work 1

Average Rates of Change

Q1 find the average rate of change of the function over the given interval or intervals

1. $f(x) = x^3 + 1$
a. (2, 3) **b.** (-1, 1)

2. $g(x) = x^2 - 2x$
a. (1, 3) **b.** (-2, 4)

3. $g(t) = 2 + \cos t$
a. $[0, \pi]$ **b.** $[-\pi, \pi]$

4. $P(\theta) = \theta^3 - 4\theta^2 + 5\theta$; [1, 2]

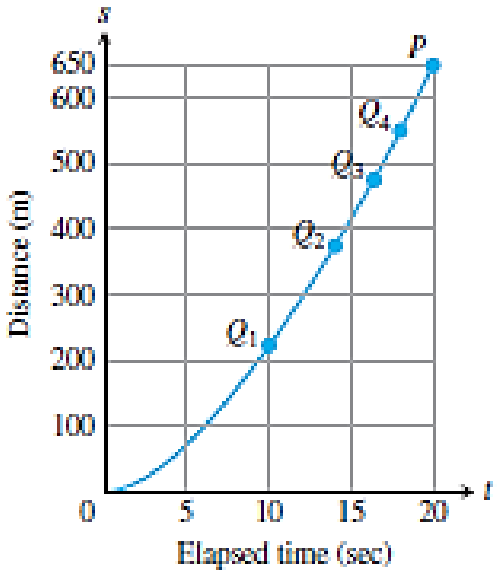
Slope of a Curve at a Point

Q2 Use the method in Example 3 to find **(a)** the slope of the curve at the given point P, and **(b)** an equation of the tangent line at P.

5. $y = x^2 - 5$, $P(2, -1)$
6. $y = 7 - x^2$, $P(2, 3)$
7. $y = x^2 - 4x$, $P(1, -3)$
8. $y = x^3 - 3x^2 + 4$, $P(2, 0)$

Instantaneous Rates of Change

9. Speed of a car The accompanying figure shows the time-to distance graph for a sports car accelerating from a standstill.



a. Estimate the slopes of secants PQ_1, PQ_2, PQ_3, and PQ_4, arranging them in order in a table like the one in Figure 2.6. What are the appropriate units for these slopes?

b. Then estimate the car's speed at time $t = 20$ sec.

10. Make a table of values for the function $F(x) = (x + 2)/(x - 2)$ at the points $x = 1.2$, $x = 11/10$, $x = 101/100$, $x = 1001/1000$, $x = 10001/10000$, and $x = 1$.

a. Find the average rate of change of $F(x)$ over the intervals $[1, x]$ for each $x \neq 1$ in your table.

b. Extending the table if necessary, try to determine the rate of change of $F(x)$ at $x = 1$.

2.2 Limit of a Function and Limit Laws

Limits of Function Values

Frequently when studying a function **y = f(x),** we find ourselves interested in the function's behavior *near* a particular point **c**, but not *at* **c**. This might be the case, for instance, if *c* is an irrational number, like **π** or $\sqrt{2}$, whose values can only be approximated by "close" rational numbers at which we actually evaluate the function instead. Another situation occurs when trying to evaluate a function at **c** leads to division by zero, which is undefined. We encountered this last circumstance when seeking the instantaneous rate of change in **y** by considering the quotient function **Δy/*h*** for **h** closer and closer to zero.

Here's a specific example in which we explore numerically how a function behaves near a particular point at which we cannot directly evaluate the function.

Example 6 How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution The given formula defines f for all real numbers x except $x = 1$ (we cannot divide by zero). For any $x \neq 1$, we can simplify the formula by factoring the numerator and canceling common factors:

$$f(x) = \frac{(x - 1)(x + 1)}{x - 1} = x + 1 \qquad \text{for} \qquad x \neq 1.$$

The graph of f is the line **$y = x + 1$** with the point **(1, 2)** *removed*. This removed point is shown as a "hole" in Figure 2.7. Even though **$f(1)$** is not defined, it is clear that we can make the value of **$f(x)$** *as close as we want* to **2** by choosing x *close enough* to **1** (Table 2.2).

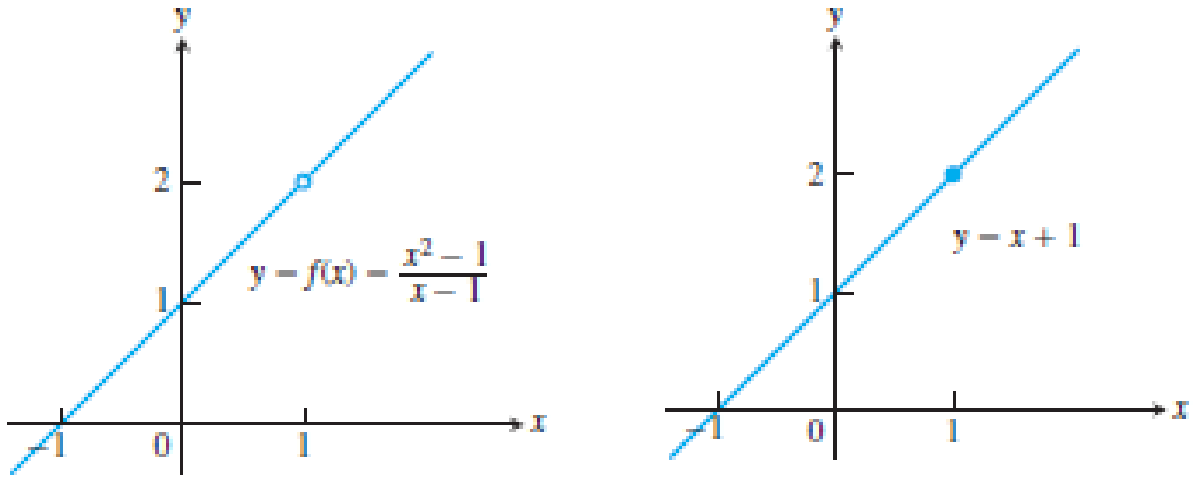


Figure 2.7 The graph of f is identical with the line $y = x + 1$ except at $x = 1$, where f is not defined (Example 1).

Generalizing the idea illustrated in Example 6, suppose $f(x)$ is defined on an open interval about c, *except possibly at c itself*. If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the **limit** L as x approaches c, and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of f(x) as x approaches c is L." For instance, in Example 1 we would say that f(x) approaches the limit 2 as x approaches 1, and write

$$\lim_{x \to 1} f(x) = 2, \qquad \text{or} \qquad \lim_{x \to 1} \frac{x^2 - 1}{x - 1} = 2.$$

Essentially, the definition says that the values of f(x) are close to the number L whenever x is close to c (on either side of c).

TABLE 2.2 As x gets closer to 1, $f(x)$ gets closer to 2.

x	$f(x) = \frac{x^2 - 1}{x - 1}$
0.9	1.9
1.1	2.1
0.99	1.99
1.01	2.01
0.999	1.999
1.001	2.001
0.999999	1.999999
1.000001	2.000001

Example 7 The limit value of a function does not depend on how the function is defined at the point being approached. Consider the three functions in Figure 2.8. The function f has limit 2 as $x \to 1$ even though f is not defined at $x = 1$. The function g has limit 2 as $x \to 1$ even though $2 \neq g(1)$. The function h is the only one of the three functions in Figure 2.8 whose limit as $x \to 1$ equals its value at $x = 1$. For h, we have $\lim_{x \to 1} h(x) = h(1)$. This equality of limit and function value is of special importance, and we return to it in Section 2.5.

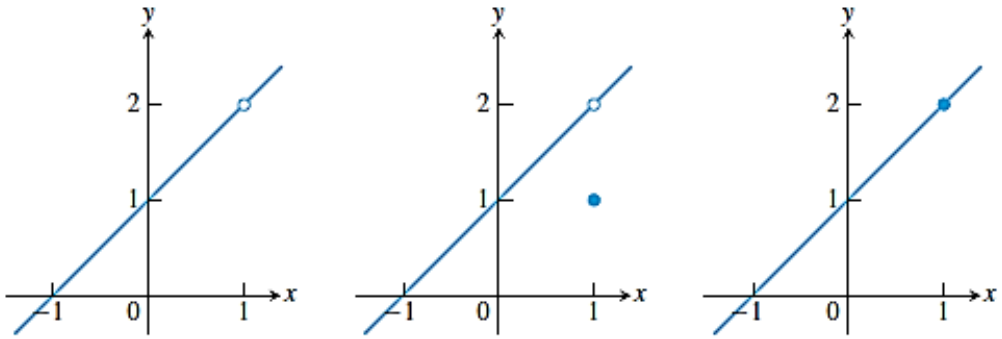


FIGURE 2.8 The limits of $f(x)$, $g(x)$, and $h(x)$ all equal 2 as x approaches 1. However, only $h(x)$ has the same function value as its limit at $x = 1$ (Example 7).

Example 8

(a) If f is the **identity function** $f(x) = x$, then for any value of c (Figure 2.9a),

$$\lim_{x \to c} f(x) = \lim_{x \to c} x = c.$$

(b) If f is the constant function $f(x) = k$ (function with the constant value k), then for any value of c (Figure 2.9b),

$$\lim_{x \to c} f(x) = \lim_{x \to c} k = k.$$

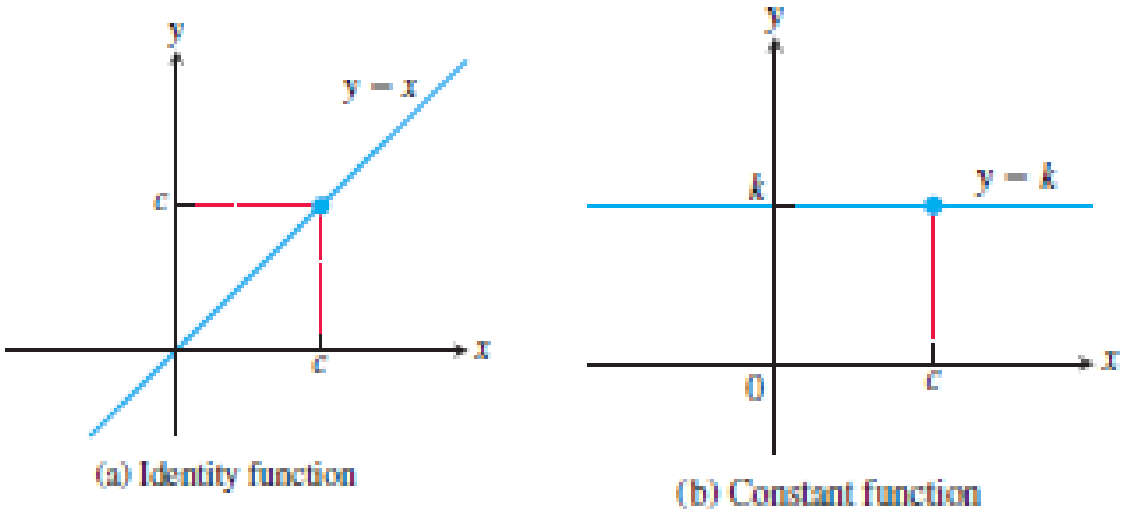


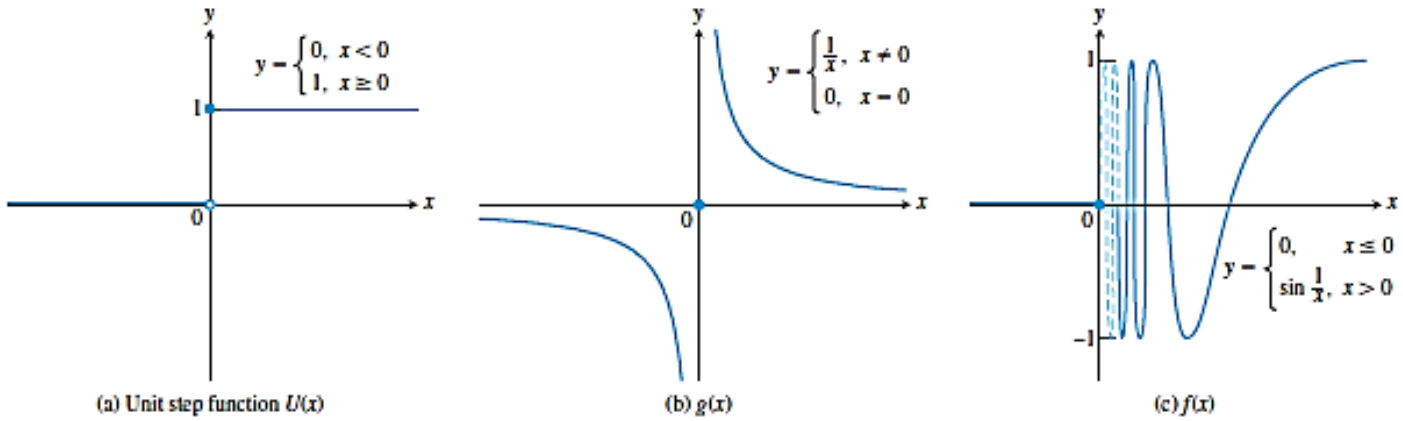
Figure 2.9 The functions in Example 3 have limits at all points c.

For instances of each of these rules we have

$$\lim_{x \to 3} x = 3 \qquad \text{and} \qquad \lim_{x \to -7} (4) = \lim_{x \to 2} (4) = 4.$$

We prove these rules in Example 8 in Section 2.3.

A function may not have a limit at a particular point. Some ways that limits can fail to exist are illustrated in Figure 2.10 and described in the next example.



(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

Figure 2.10 None of these functions has a limit as x approaches 0 (Example 9).

Example 9 Discuss the behavior of the following functions, explaining why they have no limit as x →0.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$

(b) $g(x) = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$

(c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$

Solution

(a) It *jumps*: The **unit step function** $U(x)$ has no limit as $x \to 0$ because its values jump at $x = 0$. For negative values of x arbitrarily close to zero, $U(x) = 0$. For positive values of x arbitrarily close to zero, $U(x) = 1$. There is no *single* value L approached by $U(x)$ as $x \to 0$ (Figure 2.10a).

(b) It *grows too "large" to have a limit*: $g(x)$ has no limit as $x \to 0$ because the values of g grow arbitrarily large in absolute value as $x \to 0$ and do not stay close to *any* fixed real number (Figure 2.10b). We say the function is *not bounded*.

(c) It *oscillates too much to have a limit*: $f(x)$ has no limit as $x \to 0$ because the function's values oscillate between +1 and -1 in every open interval containing 0. The values do not stay close to any one number as $x \to 0$ (Figure 2.10c).

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

THEOREM 2—Limits of Polynomials
If $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$, then

$$\lim_{x\to c} P(x) = P(c) = a_n c^n + a_{n-1}c^{n-1} + \cdots + a_0.$$

THEOREM 3—Limits of Rational Functions
If $P(x)$ and $Q(x)$ are polynomials and $Q(c) \neq 0$, then

$$\lim_{x\to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}.$$

Example 10 Use the observations $\lim_{x\to c} k = k$ and $\lim_{x\to c} x = c$ (Example 8) and the fundamental rules of limits to find the following limits.

(a) $\lim_{x\to c}(x^3 + 4x^2 - 3)$ (b) $\lim_{x\to c} \dfrac{x^4 + x^2 - 1}{x^2 + 5}$ (c) $\lim_{x\to -2} \sqrt{4x^2 - 3}$

Solution

(a) $$\lim_{x\to c}(x^3 + 4x^2 - 3) = \lim_{x\to c} x^3 + \lim_{x\to c} 4x^2 - \lim_{x\to c} 3$$ Sum and Difference Rules

$$= c^3 + 4c^2 - 3$$ Power and Multiple Rules

(b) $$\lim_{x\to c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x\to c}(x^4 + x^2 - 1)}{\lim_{x\to c}(x^2 + 5)}$$ Quotient Rule

$$= \frac{\lim_{x\to c} x^4 + \lim_{x\to c} x^2 - \lim_{x\to c} 1}{\lim_{x\to c} x^2 + \lim_{x\to c} 5}$$ Sum and Difference Rules

$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$ Power or Product Rule

(c) $$\lim_{x\to -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x\to -2}(4x^2 - 3)}$$ Root Rule with $n = 2$

$$= \sqrt{\lim_{x\to -2} 4x^2 - \lim_{x\to -2} 3}$$ Difference Rule

$$= \sqrt{4(-2)^2 - 3}$$ Product and Multiple Rules

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

Example 11 The following calculation illustrates Theorems 2and 3:

$$\lim_{x\to -1}\frac{x^3+4x^2-3}{x^2+5}=\frac{(-1)^3+4(-1)^2-3}{(-1)^2+5}=\frac{0}{6}=0$$

Example 12 Evaluate $\lim_{x\to 1}\dfrac{x^2+x-2}{x^2-x}$.

Solution We cannot substitute $x = 1$ because it makes the denominator zero. We test the numerator to see if it, too, is zero at $x = 1$. It is, so it has a factor of $(x - 1)$ in common with the denominator. Canceling this common factor gives a simpler fraction with the same values as the original for $x \neq 1$:

$$\frac{x^2+x-2}{x^2-x}=\frac{(x-1)(x+2)}{x(x-1)}=\frac{x+2}{x},\quad \text{if } x\neq 1.$$

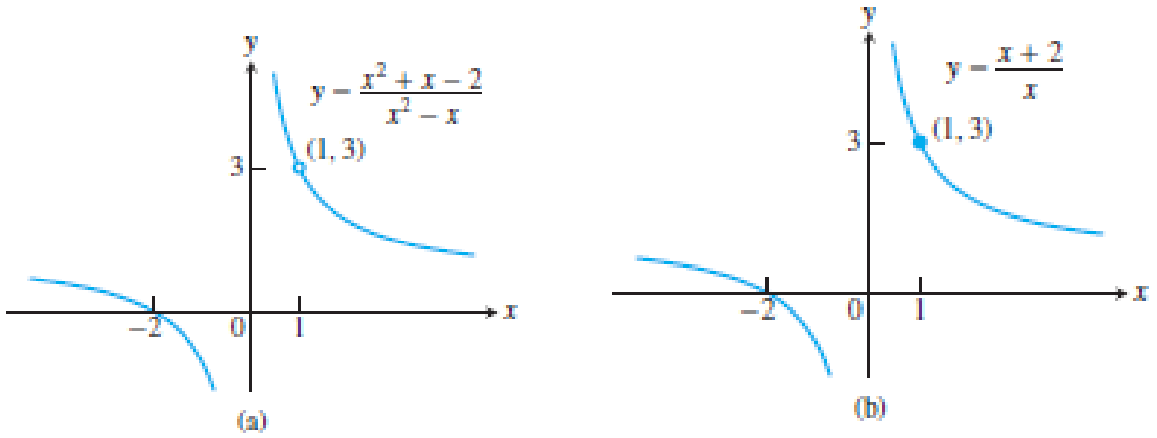
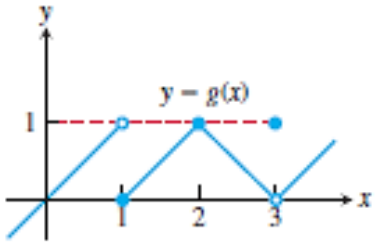


Figure 2.11 The graph of $f(x) = (x^2 + x - 2)/(x^2 - x)$ in part (a) is the same as the graph of $g(x) = (x + 2)/x$ in part (b) except at $x = 1$, where f is undefined. The functions have the same limit as $x \to 1$ (Example 12).

Home W. 2

Limits from Graphs

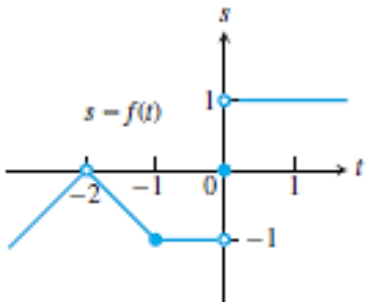
1. For the function $g(x)$ graphed here, find the following limits or explain why they do not exist.



a. $\lim_{x\to 1} g(x)$ **b.** $\lim_{x\to 2} g(x)$

c. $\lim_{x\to 3} g(x)$ **d.** $\lim_{x\to 2.5} g(x)$

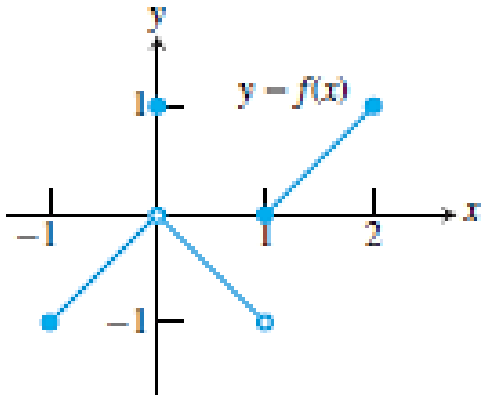
2. For the function f(t) graphed here, find the following limits or explain why they do not exist.



a. $\lim_{t\to -2} f(t)$ **b.** $\lim_{t\to -1} f(t)$

c. $\lim_{t\to 0} f(t)$ **d.** $\lim_{t\to -0.5} f(t)$

3. Which of the following statements about the function $y = f(x)$ graphed here are true, and which are false?



a. $\lim_{x\to 0} f(x)$ exists.

b. $\lim_{x\to 0} f(x) = 0$

c. $\lim_{x\to 0} f(x) = 1$

d. $\lim_{x\to 1} f(x) = 1$

e. $\lim_{x\to 1} f(x) = 0$

f. $\lim_{x\to c} f(x)$ exists at every point c in $(-1, 1)$.

g. $\lim_{x\to 1} f(x)$ does not exist.