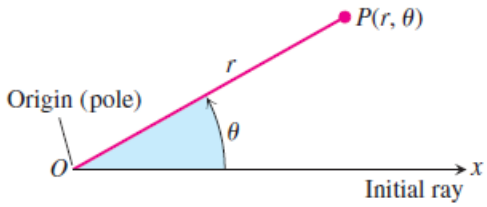
Polar coordinates

In this section we study polar coordinates and their relation to Cartesian coordinates. You will see that polar coordinates are very useful for calculating many multiple integrals.

1.1 Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (Called the **pole**) and an **initial ray** from O. Then each point P can be located by assigning to it a **polar coordinate pair** in which r gives the directed distance from O to P and gives the directed angle from the initial ray to ray OP, see the figure.



Polar Coordinates

$$P(r, \theta)$$

Directed distance from O to P

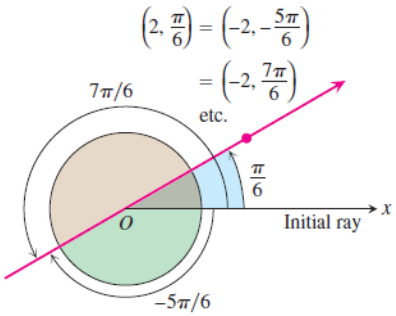
Directed angle from initial ray to OP

EXAMPLE Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 11.21). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \ldots.$$



For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \ldots.$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

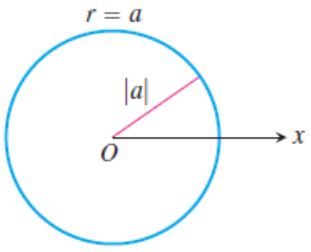
and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots.$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

1.2 Polar Equations and Graphs

Equation	Graph
$r = a$	Circle of radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray



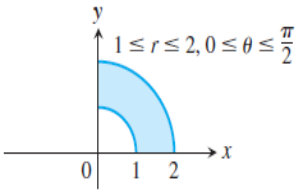
EXAMPLE Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \dfrac{\pi}{2}$

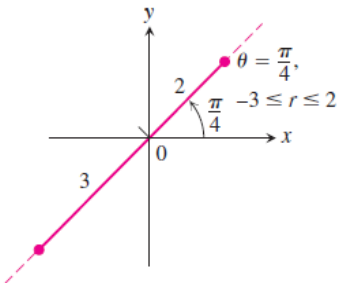
(b) $-3 \le r \le 2$ and $\theta = \dfrac{\pi}{4}$

(c) $\dfrac{2\pi}{3} \le \theta \le \dfrac{5\pi}{6}$ (no restriction on r)

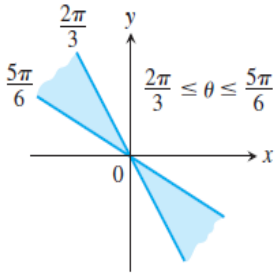
(a)



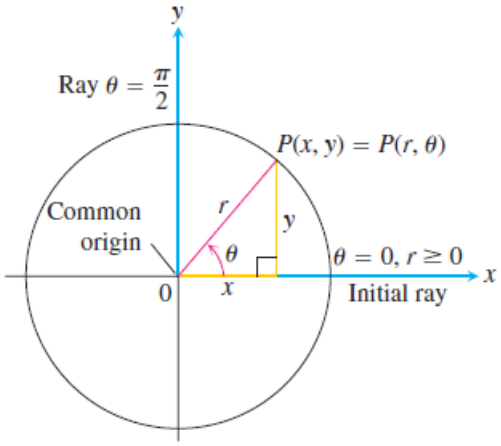
(b)



(c)



1.3 Equation relating polar and Cartesian coordinate



Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

EXAMPLE Here are some equivalent equations expressed in terms of both polar coordinates and Cartesian coordinates.

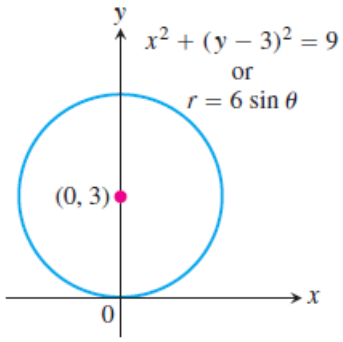
Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

Some curves are more simply expressed with polar coordinates; others are not. ■

EXAMPLE Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$

Solution We apply the equations relating polar and Cartesian coordinates:

$$\begin{aligned} x^2 + (y - 3)^2 &= 9 \\ x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y - 3)^2. \\ x^2 + y^2 - 6y &= 0 && \text{Cancellation} \\ r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2 \\ r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\ r &= 6\sin\theta && \text{Includes both possibilities} \end{aligned}$$



EXAMPLE Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$

$x = -4$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:

$$r^2 = 4r\cos\theta$$
$$x^2 + y^2 = 4x$$
$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \qquad \text{Completing the square}$$
$$(x - 2)^2 + y^2 = 4$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$r(2\cos\theta - \sin\theta) = 4$$
$$2r\cos\theta - r\sin\theta = 4$$
$$2x - y = 4$$
$$y = 2x - 4$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ■

Solve Ex11.3 Page 648 . problem 2,6,7,15,27,33,56,63

1.2. Graphing in Polar Coordinates

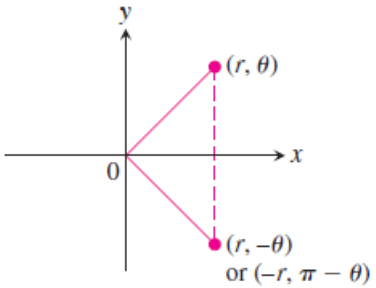
It is often helpful to have the graph of an equation in polar coordinates. This section describes techniques for graphing these equations using symmetries and tangents to the graph.

1. Symmetry

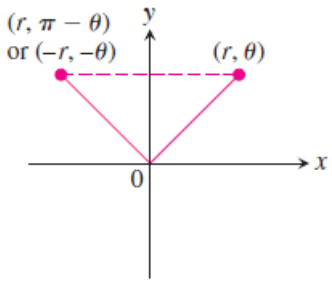
Figure **below shows** the standard polar coordinate tests for symmetry. The following summary says how the symmetric points are related.

Symmetry Tests for Polar Graphs

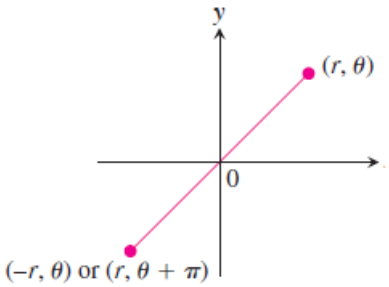
1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, then the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph **(fig a)**
2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, then the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure **b)**
3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, then the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure **c)**



(a) About the x-axis



(b) About the y-axis



(c) About the origin

2.Slope

The slope of a polar curve $r = f(\theta)$ in the xy-plane is still given by dy/dx, which is not $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad \text{Section 11.2, Eq. (1) with } t = \theta$$

$$= \frac{\dfrac{d}{d\theta}(f(\theta)\cdot\sin\theta)}{\dfrac{d}{d\theta}(f(\theta)\cdot\cos\theta)}$$

$$= \frac{\dfrac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\dfrac{df}{d\theta}\cos\theta - f(\theta)\sin\theta} \qquad \text{Product Rule for derivatives}$$

Therefore we see that dy/dx is not the same as $df/d\theta$.

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

provided $dx/d\theta \neq 0$ at (r, θ).

EXAMPLE Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$(r, \theta) \text{ on the graph} \Rightarrow r = 1 - \cos\theta$$

$$\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta)$$

$$\Rightarrow (r, -\theta) \text{ on the graph}.$$

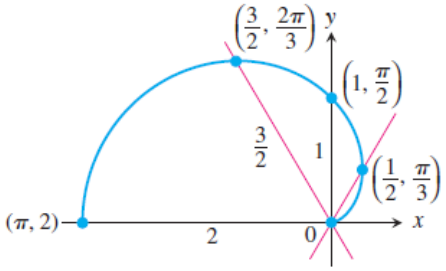
As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

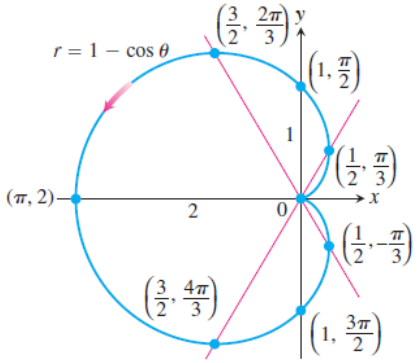
We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph (Figure 11.27). The curve is called a *cardioid* because of its heart shape. ■

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)



(c)

EXAMPLE Graph the curve $r^2 = 4\cos\theta$.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$(r, \theta) \text{ on the graph} \Rightarrow r^2 = 4\cos\theta$$
$$\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta)$$
$$\Rightarrow (r, -\theta) \text{ on the graph.}$$

The curve is also symmetric about the origin because

$$(r, \theta) \text{ on the graph} \Rightarrow r^2 = 4\cos\theta$$
$$\Rightarrow (-r)^2 = 4\cos\theta$$
$$\Rightarrow (-r, \theta) \text{ on the graph.}$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

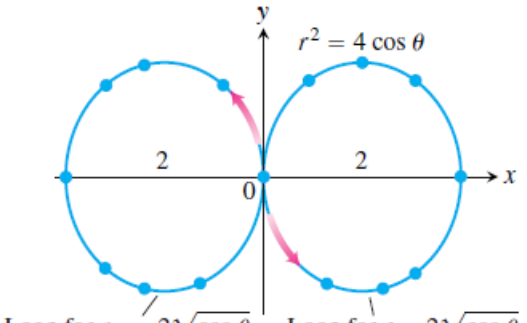
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 11.28).

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 11.28 The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded (Example 2). ■

A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ)-values, plot the corresponding points, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing that is usually quicker and more reliable is to

1. first graph $r = f(\theta)$ in the *Cartesian* $r\theta$-plane,
2. then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1– **6**. Then sketch the curves.

1. $r = 1 + \cos\theta$

2. $r = 2 - 2\cos\theta$

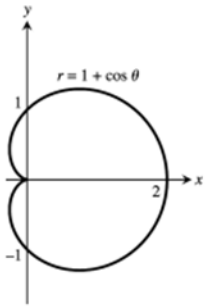
3. $r = 1 - \sin\theta$

4. $r = 1 + \sin\theta$

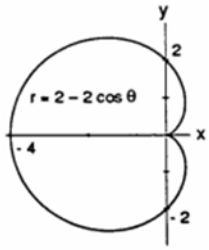
5. $r = 2 + \sin\theta$

6. $r = 1 + 2\sin\theta$

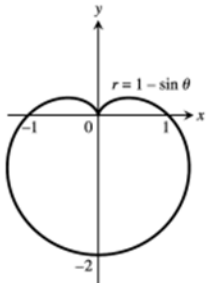
1. $1 + \cos(-\theta) = 1 + \cos\theta = r \Rightarrow$ symmetric about the x-axis; $1 + \cos(-\theta) \neq -r$ and $1 + \cos(\pi - \theta) = 1 - \cos\theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



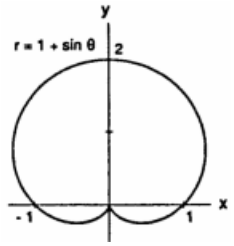
2. $2 - 2\cos(-\theta) = 2 - 2\cos\theta = r \Rightarrow$ symmetric about the x-axis; $2 - 2\cos(-\theta) \neq -r$ and $2 - 2\cos(\pi - \theta) = 2 + 2\cos\theta \neq r \Rightarrow$ not symmetric about the y-axis; therefore not symmetric about the origin



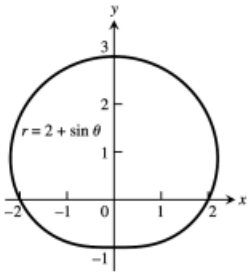
3. $1 - \sin(-\theta) = 1 + \sin\theta \neq r$ and $1 - \sin(\pi - \theta) = 1 - \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis; $1 - \sin(\pi - \theta) = 1 - \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



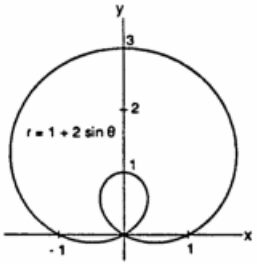
4. $1 + \sin(-\theta) = 1 - \sin\theta \neq r$ and $1 + \sin(\pi - \theta)$
$= 1 + \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis;
$1 + \sin(\pi - \theta) = 1 + \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



5. $2 + \sin(-\theta) = 2 - \sin\theta \neq r$ and $2 + \sin(\pi - \theta)$
$= 2 + \sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis;
$2 + \sin(\pi - \theta) = 2 + \sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin

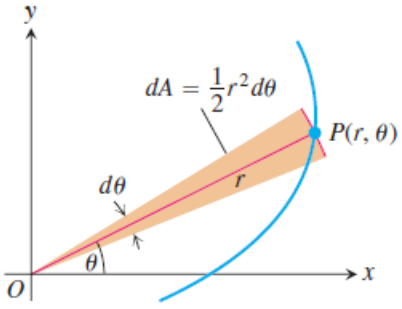


6. $1 + 2\sin(-\theta) = 1 - 2\sin\theta \neq r$ and $1 + 2\sin(\pi - \theta)$
$= 1 + 2\sin\theta \neq -r \Rightarrow$ not symmetric about the x-axis;
$1 + 2\sin(\pi - \theta) = 1 + 2\sin\theta = r \Rightarrow$ symmetric about the y-axis; therefore not symmetric about the origin



Solve Ex 11.4 Page 653 problem 7 to 12,13,14,18,22,25

1.3. Areas and Lengths in Polar Coordinates



The area of the sector (dθ) $is\ = \frac{\pi r^2}{2\pi}.d\theta = \frac{1}{2}r^2 d\theta$

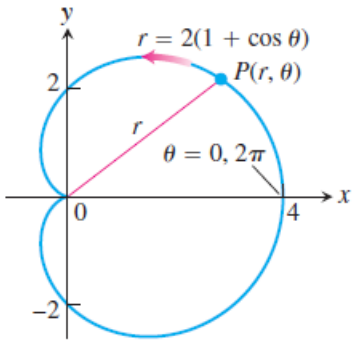
Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Figure)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \big(f(\theta)\big)^2 \, d\theta.$$

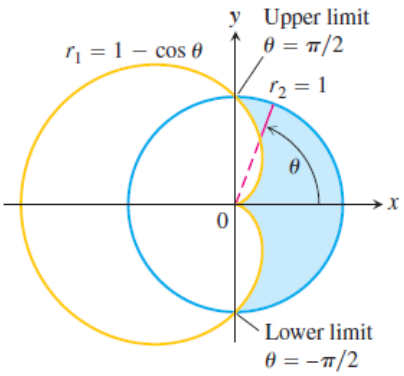
EXAMPLE Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.



$$\begin{aligned}
\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta &= \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta \\
&= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta) \, d\theta \\
&= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\frac{1 + \cos 2\theta}{2}\right) d\theta \\
&= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta) \, d\theta \\
&= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.
\end{aligned}$$

■

EXAMPLE Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.



$$A = \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta$$

$$= 2\int_0^{\pi/2} \frac{1}{2}\left(r_2^2 - r_1^2\right) d\theta \qquad \text{Symmetry}$$

$$= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \qquad \text{Square } r_1.$$

$$= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2} \left(2\cos\theta - \frac{1 + \cos 2\theta}{2}\right) d\theta$$

$$= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.$$

1.4 Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta.$$

The parametric length formula,

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

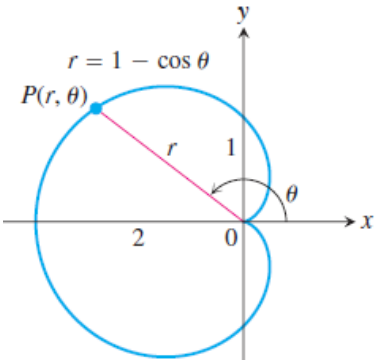
Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta.$$

EXAMPLE Find the length of the cardioid $r = 1 - \cos\theta$.

Solution We sketch the cardioid to determine the limits of integration (Figure). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.



With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

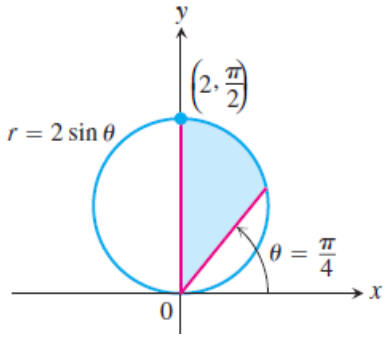
we have

$$r^2 + \left(\frac{dr}{d\theta}\right)^2 = (1 - \cos\theta)^2 + (\sin\theta)^2$$
$$= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta$$

and

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$
$$= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2(\theta/2)$$
$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$
$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin(\theta/2) \geq 0 \quad \text{for} \quad 0 \leq \theta \leq 2\pi$$
$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$

Example: Find the areas of the regions bounded by the circle $r = 2sin\theta$ for $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$ also find the length of the curve in that region.



$$A = \int_{\pi/4}^{\pi/2} \tfrac{1}{2}(2\sin\theta)^2 \, d\theta = 2\int_{\pi/4}^{\pi/2} \sin^2\theta \, d\theta = 2\int_{\pi/4}^{\pi/2} \tfrac{1-\cos 2\theta}{2} \, d\theta = \int_{\pi/4}^{\pi/2} (1-\cos 2\theta) d\theta = \left[\theta - \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/2}$$
$$= \left(\tfrac{\pi}{2} - 0\right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{4} + \tfrac{1}{2}$$

$$L = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sqrt{(2\sin\theta)^2 + (2\cos\theta)^2} d\theta = \frac{\pi}{2}$$

Ex 11.5 P656. Problem 1,3 to 16, 21 to 28