1.Integration

The *integral* is of fundamental importance in statistics, the sciences, and engineering. We use it to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a dam's floodgates. We focus on the integral concept and its use in computing areas of various regions with curved boundaries.. In this chapter we study a number of other important techniques for finding antiderivatives (or indefinite integrals) for many combinations of functions whose antiderivatives cannot be found using the methods presented before

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

2.Techniques of Integration

2.1. Trigonometric Integrals

Some of the following trigonometry identities may be needed.

- A.) $\cos^2 x + \sin^2 x = 1$
- B.) $\sin 2x = 2\sin x \cos x$
- C.) $\cos 2x = 2\cos^2 x - 1$ so that $\cos^2 x = \dfrac{1+\cos 2x}{2}$
- D.) $\cos 2x = 1 - 2\sin^2 x$ so that $\sin^2 x = \dfrac{1-\cos 2x}{2}$
- E.) $\cos 2x = \cos^2 x - \sin^2 x$
- F.) $1 + \tan^2 x = \sec^2 x$ so that $\tan^2 x = \sec^2 x - 1$
- G.) $1 + \cot^2 x = \csc^2 x$ so that $\cot^2 x = \csc^2 x - 1$

2.1.1 The direct integration for trig. Functions

- 1.) $\int \cos x \; dx = \sin x + C$
- 2.) $\int \sin x \; dx = -\cos x + C$
- 3.) $\int \sec^2 x \; dx = \tan x + C$
- 4.) $\int \csc^2 x \; dx = -\cot x + C$
- 5.) $\int \sec x \tan x \; dx = \sec x + C$
- 6.) $\int \csc x \cot x \; dx = -\csc x + C$
- 7.) $\int \tan x \; dx = \ln|\sec x| + C$
- 8.) $\int \cot x \; dx = \ln|\sin x| + C$
- 9.) $\int \sec x \; dx = \ln|\sec x + \tan x| + C$
- 10.) $\int \csc x \; dx = \ln|\csc x - \cot x| + C$

Solve 1. $\int e^x \sin e^x \, dx$, 2. $\int \cos^2 x dx$, 3. $\int \tan^2 x dx$,

4. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\, dx$

5. $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\, dx$

(*Hint*: Multiply by $\sqrt{\dfrac{1-\sin x}{1-\sin x}}$.)

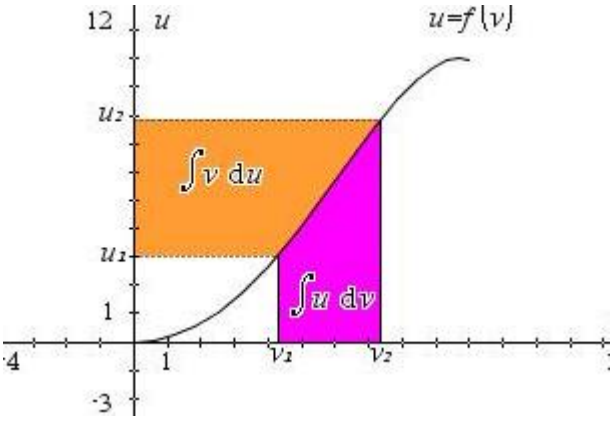
4. $\int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\,dx = \int_{\pi/3}^{\pi/2} \frac{\sin^2 x}{\sqrt{1-\cos x}}\frac{\sqrt{1+\cos x}}{\sqrt{1+\cos x}}\,dx = \int_{\pi/3}^{\pi/2} \frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{1-\cos^2 x}}\,dx = \int_{\pi/3}^{\pi/2} \frac{\sin^2 x\sqrt{1+\cos x}}{\sqrt{\sin^2 x}}\,dx$
$= \int_{\pi/3}^{\pi/2} \sin x\sqrt{1+\cos x}\,dx = \left[-\frac{2}{3}(1+\cos x)^{3/2}\right]_{\pi/3}^{\pi/2} = -\frac{2}{3}\left(1+\cos\left(\frac{\pi}{2}\right)\right)^{3/2} + \frac{2}{3}\left(1+\cos\left(\frac{\pi}{3}\right)\right)^{3/2} = -\frac{2}{3} + \frac{2}{3}\left(\frac{3}{2}\right)^{3/2}$
$= \sqrt{\frac{3}{2}} - \frac{2}{3}$

5. $\int_0^{\pi/6} \sqrt{1+\sin x}\,dx = \int_0^{\pi/6} \frac{\sqrt{1+\sin x}}{1}\frac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6} \frac{\sqrt{1-\sin^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6} \frac{\sqrt{\cos^2 x}}{\sqrt{1-\sin x}}\,dx = \int_0^{\pi/6} \frac{\cos x}{\sqrt{1-\sin x}}\,dx$
$= \left[-2(1-\sin x)^{1/2}\right]_0^{\pi/6} = -2\sqrt{1-\sin\left(\frac{\pi}{6}\right)} + 2\sqrt{1-\sin 0} = -2\sqrt{\frac{1}{2}} + 2\sqrt{1} = 2 - \sqrt{2}$

2.2. Integration by Parts

Integration by parts is used to integrate the product of two or more functions. The two functions are u and v. Among the two functions, the first function u is selected such that its derivative formula exists, and the second function v is chosen such that an integral of such a function exists. It is a technique for simplifying integrals of the form

$$\int u\,dv$$



then following the above drawing, it can be written as

$$\int u\,dv = uv - \int v du$$

EXAMPLE 1 Find

$$\int x \cos x \, dx.$$

Solution We use the formula $\int u \, dv = uv - \int v \, du$ with

$$u = x, \qquad dv = \cos x \, dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx = x \sin x + \cos x + C. \quad \blacksquare$$

EXAMPLE 2 Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\qquad v = x.$ Simplest antiderivative

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

Example 3 Evaluate the following integral.

$$\int x\mathbf{e}^{6x} \, dx$$

$$u = x \qquad dv = \mathbf{e}^{6x} \, dx$$

$$du = dx \qquad v = \int \mathbf{e}^{6x} \, dx = \frac{1}{6}\mathbf{e}^{6x}$$

The integral is then,

$$\int x\mathbf{e}^{6x} \, dx = \frac{x}{6}\mathbf{e}^{6x} - \int \frac{1}{6}\mathbf{e}^{6x} \, dx$$
$$= \frac{x}{6}\mathbf{e}^{6x} - \frac{1}{36}\mathbf{e}^{6x} + c$$

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2.3. Integration by Partial fraction

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x)=\frac{P(x)}{Q(x)}$$

where both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}, \quad k=1,2,3,\ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad k=1,2,3,\ldots$

Example Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}\,dx$$

Solution

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A}{x-3} + \frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)} = \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose *A* and *B* so that the numerators of these two are equal for every *x*. To do this we'll need to set the numerators equal.

$$3x+11 = A(x+2)+B(x-3)$$

$$x=-2: \qquad 5 = A(0)+B(-5) \quad \Rightarrow \quad B=-1$$

$$x=3 \quad :20 = A(5)+B(0) \quad \Rightarrow \quad A=4$$

Then

$$\begin{aligned}\int \frac{3x+11}{x^2-x-6}\,dx &= \int \frac{4}{x-3} - \frac{1}{x+2}\,dx \\ &= \int \frac{4}{x-3}\,dx - \int \frac{1}{x+2}\,dx \\ &= 4\ln|x-3| - \ln|x+2| + c\end{aligned}$$

EXAMPLE Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx.$$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\begin{aligned} \int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\,dx &= \int \left[\frac{3}{4}\frac{1}{x - 1} + \frac{1}{2}\frac{1}{x + 1} - \frac{1}{4}\frac{1}{x + 3}\right] dx \\ &= \frac{3}{4}\ln|x - 1| + \frac{1}{2}\ln|x + 1| - \frac{1}{4}\ln|x + 3| + K, \end{aligned}$$

EXAMPLE Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\begin{aligned} \frac{6x + 7}{(x + 2)^2} &= \frac{A}{x + 2} + \frac{B}{(x + 2)^2} \\ 6x + 7 &= A(x + 2) + B \qquad \text{Multiply both sides by } (x + 2)^2. \\ &= Ax + (2A + B) \end{aligned}$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\begin{aligned} \int \frac{6x + 7}{(x + 2)^2}\,dx &= \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2}\right) dx \\ &= 6\int \frac{dx}{x + 2} - 5\int (x + 2)^{-2}\,dx \\ &= 6\ln|x + 2| + 5(x + 2)^{-1} + C. \end{aligned}$$

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