Evaluate the integrals

51. $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\, dx$ **52.** $\displaystyle\int_{0}^{\ln 2} \tanh 2x\, dx$

53. $\displaystyle\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\, d\theta$ **54.** $\displaystyle\int_{0}^{\ln 2} 4e^{-\theta}\sinh\theta\, d\theta$

51. $\displaystyle\int_{\ln 2}^{\ln 4}\coth x\,dx = \int_{\ln 2}^{\ln 4}\frac{\cosh x}{\sinh x}\,dx = \int_{3/4}^{15/8}\frac{1}{u}\,du = [\ln|u|]_{3/4}^{15/8} = \ln\left|\frac{15}{8}\right| - \ln\left|\frac{3}{4}\right| = \ln\left|\frac{15}{8}\cdot\frac{4}{3}\right| = \ln\frac{5}{2}$,
where $u = \sinh x$, $du = \cosh x\,dx$, the lower limit is $\sinh(\ln 2) = \frac{e^{\ln 2} - e^{-\ln 2}}{2} = \frac{2-\left(\frac{1}{2}\right)}{2} = \frac{3}{4}$ and the upper limit is $\sinh(\ln 4) = \frac{e^{\ln 4} - e^{-\ln 4}}{2} = \frac{4-\left(\frac{1}{4}\right)}{2} = \frac{15}{8}$

52. $\displaystyle\int_{0}^{\ln 2}\tanh 2x\,dx = \int_{0}^{\ln 2}\frac{\sinh 2x}{\cosh 2x}\,dx = \frac{1}{2}\int_{1}^{17/8}\frac{1}{u}\,du = \frac{1}{2}[\ln|u|]_{1}^{17/8} = \frac{1}{2}\left[\ln\left(\frac{17}{8}\right) - \ln 1\right] = \frac{1}{2}\ln\frac{17}{8}$, where
$u = \cosh 2x$, $du = 2\sinh(2x)\,dx$, the lower limit is $\cosh 0 = 1$ and the upper limit is $\cosh(2\ln 2) = \cosh(\ln 4)$
$= \frac{e^{\ln 4} + e^{-\ln 4}}{2} = \frac{4+\left(\frac{1}{4}\right)}{2} = \frac{17}{8}$

53. $\displaystyle\int_{-\ln 4}^{-\ln 2} 2e^{\theta}\cosh\theta\,d\theta = \int_{-\ln 4}^{-\ln 2} 2e^{\theta}\left(\frac{e^{\theta}+e^{-\theta}}{2}\right)d\theta = \int_{-\ln 4}^{-\ln 2}(e^{2\theta}+1)\,d\theta = \left[\frac{e^{2\theta}}{2} + \theta\right]_{-\ln 4}^{-\ln 2}$
$= \left(\frac{e^{-2\ln 2}}{2} - \ln 2\right) - \left(\frac{e^{-2\ln 4}}{2} - \ln 4\right) = \left(\frac{1}{8} - \ln 2\right) - \left(\frac{1}{32} - \ln 4\right) = \frac{3}{32} - \ln 2 + 2\ln 2 = \frac{3}{32} + \ln 2$

54. $\displaystyle\int_{0}^{\ln 2} 4e^{-\theta}\sinh\theta\,d\theta = \int_{0}^{\ln 2} 4e^{-\theta}\left(\frac{e^{\theta}-e^{-\theta}}{2}\right)d\theta = 2\int_{0}^{\ln 2}(1-e^{-2\theta})\,d\theta = 2\left[\theta + \frac{e^{-2\theta}}{2}\right]_{0}^{\ln 2}$
$= 2\left[\left(\ln 2 + \frac{e^{-2\ln 2}}{2}\right) - \left(0 + \frac{e^{0}}{2}\right)\right] = 2\left(\ln 2 + \frac{1}{8} - \frac{1}{2}\right) = 2\ln 2 + \frac{1}{4} - 1 = \ln 4 - \frac{3}{4}$

60. $\displaystyle\int_{0}^{\ln 10} 4\sinh^2\left(\frac{x}{2}\right)dx$

$\displaystyle\int_{0}^{\ln 10} 4\sinh^2\left(\tfrac{x}{2}\right)dx = \int_{0}^{\ln 10} 4\left(\frac{\cosh x - 1}{2}\right)dx = 2\int_{0}^{\ln 10}(\cosh x - 1)\,dx = 2[\sinh x - x]_{0}^{\ln 10}$
$= 2[(\sinh(\ln 10) - \ln 10) - (\sinh 0 - 0)] = e^{\ln 10} - e^{-\ln 10} - 2\ln 10 = 10 - \frac{1}{10} - 2\ln 10 = 9.9 - 2\ln 10$

67. $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4 + x^2}}$

68. $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1 + 9x^2}}$

69. $\displaystyle\int_{5/4}^{2} \frac{dx}{1 - x^2}$

70. $\displaystyle\int_0^{1/2} \frac{dx}{1 - x^2}$

71. $\displaystyle\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1 - 16x^2}}$

72. $\displaystyle\int_1^2 \frac{dx}{x\sqrt{4 + x^2}}$

73. $\displaystyle\int_0^{\pi} \frac{\cos x\,dx}{\sqrt{1 + \sin^2 x}}$

74. $\displaystyle\int_1^e \frac{dx}{x\sqrt{1 + (\ln x)^2}}$

67. (a) $\displaystyle\int_0^{2\sqrt{3}} \frac{dx}{\sqrt{4+x^2}} = \left[\sinh^{-1}\frac{x}{2}\right]_0^{2\sqrt{3}} = \sinh^{-1}\sqrt{3} - \sinh 0 = \sinh^{-1}\sqrt{3}$

68. (a) $\displaystyle\int_0^{1/3} \frac{6\,dx}{\sqrt{1+9x^2}} = 2\int_0^1 \frac{dx}{\sqrt{a^2+u^2}}$, where $u = 3x$, $du = 3\,dx$, $a = 1$

$= [2\sinh^{-1} u]_0^1 = 2(\sinh^{-1} 1 - \sinh^{-1} 0) = 2\sinh^{-1} 1$

69. (a) $\displaystyle\int_{5/4}^{2} \frac{1}{1-x^2}\,dx = [\coth^{-1} x]_{5/4}^{2} = \coth^{-1} 2 - \coth^{-1}\frac{5}{4}$

70. (a) $\displaystyle\int_0^{1/2} \frac{1}{1-x^2}\,dx = [\tanh^{-1} x]_0^{1/2} = \tanh^{-1}\frac{1}{2} - \tanh^{-1} 0 = \tanh^{-1}\frac{1}{2}$

71. (a) $\displaystyle\int_{1/5}^{3/13} \frac{dx}{x\sqrt{1-16x^2}} = \int_{4/5}^{12/13} \frac{du}{u\sqrt{a^2-u^2}}$, where $u = 4x$, $du = 4\,dx$, $a = 1$

$= [-\operatorname{sech}^{-1} u]_{\cdot\cdot}^{12/13} = -\operatorname{sech}^{-1}\frac{12}{\cdot\cdot} + \operatorname{sech}^{-1}\frac{4}{\cdot}$

72. (a) $\displaystyle\int_1^2 \frac{dx}{x\sqrt{4+x^2}} = \left[-\frac{1}{2}\operatorname{csch}^{-1}\left|\frac{x}{2}\right|\right]_1^2 = -\frac{1}{2}\left(\operatorname{csch}^{-1} 1 - \operatorname{csch}^{-1}\frac{1}{2}\right) = \frac{1}{2}\left(\operatorname{csch}^{-1}\frac{1}{2} - \operatorname{csch}^{-1} 1\right)$

73. (a) $\displaystyle\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^0 \frac{1}{\sqrt{1+u^2}}\,du = [\sinh^{-1} u]_0^0 = \sinh^{-1} 0 - \sinh^{-1} 0 = 0$, where $u = \sin x$, $du = \cos x\,dx$

74. (a) $\displaystyle\int_1^e \frac{dx}{x\sqrt{1+(\ln x)^2}} = \int_0^1 \frac{du}{\sqrt{a^2+u^2}}$, where $u = \ln x$, $du = \frac{1}{x}\,dx$, $a = 1$

$= [\sinh^{-1} u]_0^1 = \sinh^{-1} 1 - \sinh^{-1} 0 = \sinh^{-1} 1$