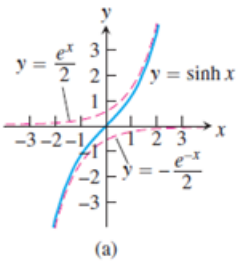
1.Transcendental Functions

These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions as well.

1.1 Hyperbolic Functions

The hyperbolic functions are formed by taking combinations of the two exponential functions e^x *and* e^{-x}. The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical applications.

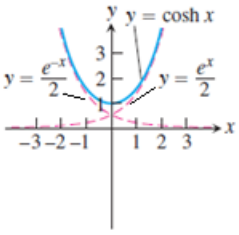
TABLE 1 The six basic hyperbolic functions



(a)

Hyperbolic sine:

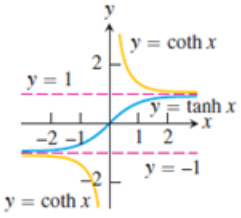
$$\sinh x = \frac{e^x - e^{-x}}{2}$$



(b)

Hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2}$$



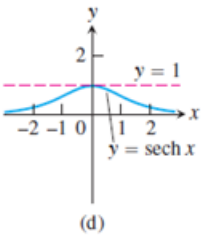
(c)

Hyperbolic tangent:

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Hyperbolic cotangent:

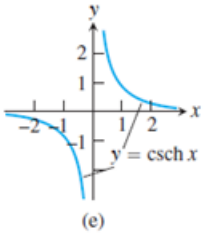
$$\coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$



(d)

Hyperbolic secant:

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$



(e)

Hyperbolic cosecant:

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

TABLE 2 Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$
$$\sinh 2x = 2 \sinh x \cosh x$$
$$\cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$
$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$
$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$
$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

Derivatives and Integrals of Hyperbolic Functions

The six hyperbolic functions, being rational combinations of the differentiable functions e^x and e^{-x}, have derivatives at every point at which they are defined (Table **1**). Again, there are similarities with trigonometric functions.

The derivative formulas are derived from the derivative of e^u:

$$\begin{aligned}
\frac{d}{dx}(\sinh u) &= \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right) && \text{Definition of sinh } u \\
&= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2} && \text{Derivative of } e^u \\
&= \cosh u \frac{du}{dx}. && \text{Definition of cosh } u
\end{aligned}$$

This gives the first derivative formula. From the definition, we can calculate the derivative of the hyperbolic cosecant function, as follows:

$$\begin{aligned}
\frac{d}{dx}(\operatorname{csch} u) &= \frac{d}{dx}\left(\frac{1}{\sinh u}\right) && \text{Definition of csch } u \\
&= -\frac{\cosh u}{\sinh^2 u}\frac{du}{dx} && \text{Quotient Rule} \\
&= -\frac{1}{\sinh u}\frac{\cosh u}{\sinh u}\frac{du}{dx} && \text{Rearrange terms.} \\
&= -\operatorname{csch} u \coth u \frac{du}{dx} && \text{Definitions of csch } u \text{ and coth } u
\end{aligned}$$

The other formulas in Table **2** are obtained similarly.

The derivative formulas lead to the integral formulas in Table **3** .

TABLE 2a Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE 3 Integral formulas for hyperbolic functions

$$\int \sinh u \, du = \cosh u + C$$

$$\int \cosh u \, du = \sinh u + C$$

$$\int \operatorname{sech}^2 u \, du = \tanh u + C$$

$$\int \operatorname{csch}^2 u \, du = -\coth u + C$$

$$\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$$

$$\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$$

EXAMPLE

(a) $\dfrac{d}{dt}\left(\tanh\sqrt{1+t^2}\right) = \operatorname{sech}^2\sqrt{1+t^2}\cdot\dfrac{d}{dt}\left(\sqrt{1+t^2}\right)$

$$= \frac{t}{\sqrt{1+t^2}}\operatorname{sech}^2\sqrt{1+t^2}$$

(b) $\displaystyle\int \coth 5x\,dx = \int \frac{\cosh 5x}{\sinh 5x}\,dx = \frac{1}{5}\int \frac{du}{u}$ $\quad u = \sinh 5x,\ du = 5\cosh 5x\,dx$

$$= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sinh 5x| + C$$

(c) $\displaystyle\int_0^1 \sinh^2 x\,dx = \int_0^1 \frac{\cosh 2x - 1}{2}\,dx$ $\quad$ Table 7.4

$$= \frac{1}{2}\int_0^1 (\cosh 2x - 1)\,dx = \frac{1}{2}\left[\frac{\sinh 2x}{2} - x\right]_0^1$$

$$= \frac{\sinh 2}{4} - \frac{1}{2} \approx 0.40672$$ $\quad$ Evaluate with a calculator.

(d) $\displaystyle\int_0^{\ln 2} 4e^x \sinh x\,dx = \int_0^{\ln 2} 4e^x\,\frac{e^x - e^{-x}}{2}\,dx = \int_0^{\ln 2} (2e^{2x} - 2)\,dx$

$$= \left[e^{2x} - 2x\right]_0^{\ln 2} = (e^{2\ln 2} - 2\ln 2) - (1 - 0)$$

$$= 4 - 2\ln 2 - 1 \approx 1.6137$$ ■

Inverse Hyperbolic Functions and their derivatives

TABLE 4 Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1}u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1}u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1}u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1}u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1}u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1}u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \qquad u \neq 0$$

EXAMPLE Show that if u is a differentiable function of x whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}.$$

Let $cosh^{-1}x = y$

Then $x = cosh\, y$ Differentiate with respect to x then

$$1 = \sinh y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{sinhy} = \frac{1}{\sinh(cosh^{-1}x)}$$ but $cosh^2x - sinh^2x = 1$

$$\frac{dy}{dx} = \frac{1}{sinhy} = \frac{1}{\sinh(cosh^{-1}x)} = \frac{1}{\sqrt{cosh^2(cosh^{-1}x)-1}} = \frac{1}{\sqrt{cosh^2y-1}} = \frac{1}{\sqrt{x^2-1}}$$

then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}.$$

Homework:

Use the definitions of cosh x and sinh x to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

Finding Derivatives

In Exercises 13–24, find the derivative of y with respect to the appropriate variable.

13. $y = 6\sinh\frac{x}{3}$ **14.** $y = \frac{1}{2}\sinh(2x + 1)$

15. $y = 2\sqrt{t}\tanh\sqrt{t}$ **16.** $y = t^2\tanh\frac{1}{t}$

17. $y = \ln(\sinh z)$ **18.** $y = \ln(\cosh z)$

In Exercises below find the derivative of y with respect to the appropriate variable.

25. $y = \sinh^{-1}\sqrt{x}$ **26.** $y = \cosh^{-1} 2\sqrt{x + 1}$

27. $y = (1 - \theta)\tanh^{-1}\theta$ **28.** $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

29. $y = (1 - t)\coth^{-1}\sqrt{t}$ **30.** $y = (1 - t^2)\coth^{-1} t$

TABLE 5 Integrals leading to inverse hyperbolic functions

1. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \sinh^{-1}\left(\frac{u}{a}\right) + C, \qquad a > 0$

2. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \cosh^{-1}\left(\frac{u}{a}\right) + C, \qquad u > a > 0$

3. $\displaystyle\int \frac{du}{a^2-u^2} = \begin{cases} \frac{1}{a}\tanh^{-1}\left(\frac{u}{a}\right) + C, & u^2 < a^2 \\ \frac{1}{a}\coth^{-1}\left(\frac{u}{a}\right) + C, & u^2 > a^2 \end{cases}$

4. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{u}{a}\right) + C, \qquad 0 < u < a$

5. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{u}{a}\right| + C, \qquad u \neq 0 \text{ and } a > 0$

EXAMPLE Evaluate

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}}.$$

Solution The indefinite integral is

$$\int \frac{2\,dx}{\sqrt{3+4x^2}} = \int \frac{du}{\sqrt{a^2+u^2}} \qquad u = 2x, \quad du = 2\,dx, \quad a = \sqrt{3}$$

$$= \sinh^{-1}\left(\frac{u}{a}\right) + C \qquad \text{Formula from Table 7.9}$$

$$= \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right) + C.$$

Therefore,

$$\int_0^1 \frac{2\,dx}{\sqrt{3+4x^2}} = \sinh^{-1}\left(\frac{2x}{\sqrt{3}}\right)\Big]_0^1 = \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - \sinh^{-1}(0)$$

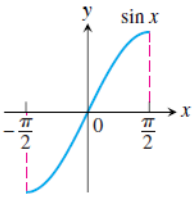
$$= \sinh^{-1}\left(\frac{2}{\sqrt{3}}\right) - 0 \approx 0.98665. \qquad \blacksquare$$

Ex 7.3 Pages 442. Problem 13 to 24, 37 to 40, 67 to 74

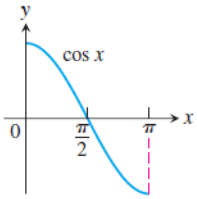
2. **Inverse Trigonometric Functions**

The six basic trigonometric functions of a general radian angle x was previously reviewed. These functions are not one-to-one (their values repeat periodically). However, we can restrict their domains to intervals on which they are one-to-one function as follow

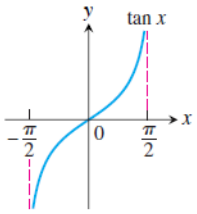
Domain restrictions that make the trigonometric functions one-to-one



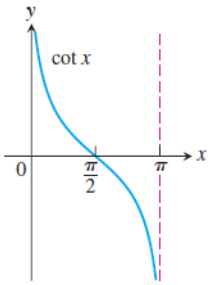
$y = \sin x$
Domain: $[-\pi/2, \pi/2]$
Range: $[-1, 1]$



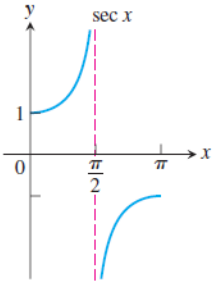
$y = \cos x$
Domain: $[0, \pi]$
Range: $[-1, 1]$



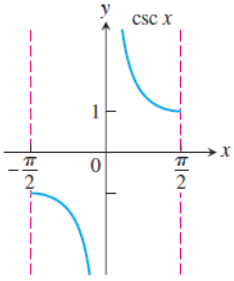
$y = \tan x$
Domain: $(-\pi/2, \pi/2)$
Range: $(-\infty, \infty)$



$y = \cot x$
Domain: $(0, \pi)$
Range: $(-\infty, \infty)$

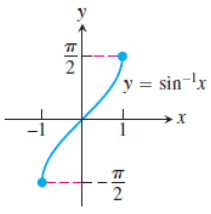


$y = \sec x$
Domain: $[0, \pi/2) \cup (\pi/2, \pi]$
Range: $(-\infty, -1] \cup [1, \infty)$



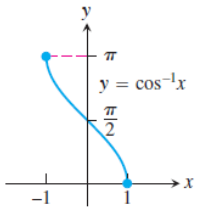
$y = \csc x$
Domain: $[-\pi/2, 0) \cup (0, \pi/2]$
Range: $(-\infty, -1] \cup [1, \infty)$

Domain: $-1 \le x \le 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$



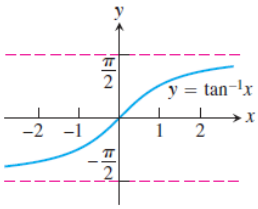
(a)

Domain: $-1 \le x \le 1$
Range: $0 \le y \le \pi$



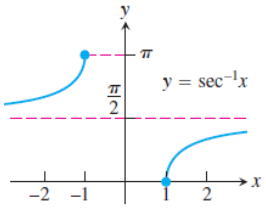
(b)

Domain: $-\infty < x < \infty$
Range: $-\frac{\pi}{2} < y < \frac{\pi}{2}$



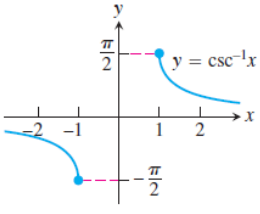
(c)

Domain: $x \le -1$ or $x \ge 1$
Range: $0 \le y \le \pi, y \ne \frac{\pi}{2}$



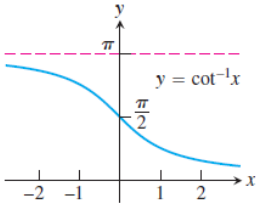
(d)

Domain: $x \le -1$ or $x \ge 1$
Range: $-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$



(e)

Domain: $-\infty < x < \infty$
Range: $0 < y < \pi$



(f)

2.2 The derivatives of the inverse trigonometric functions

1) $y = cos^{-1}x$

$$x = cosy$$

$$dx = -siny\, dy$$

$$\frac{dy}{dx} = -\frac{1}{siny} = -\frac{1}{\sqrt{1-cos^2y}} = -\frac{1}{\sqrt{1-x^2}}$$

If u is a differential function of x Then

$$\frac{dy}{dx} = -\frac{du/dx}{\sqrt{1-u^2}}$$

2) $y = sin^{-1}x$

$$x = siny$$

$$dx = cosy\, dy$$

$$\frac{dy}{dx} = \frac{1}{cosy} = \frac{1}{\sqrt{1-sin^2y}} = \frac{1}{\sqrt{1-x^2}}$$

If u is a differential function of x Then

$$\frac{dy}{dx} = \frac{du/dx}{\sqrt{1-u^2}}$$

2) $y = tan^{-1}x$

$$x = tany = \frac{siny}{cosy}$$

$$1 = \frac{cos^2yy' + sin^2yy'}{cos^2y}$$

$$y' = cos^2y$$

But $cosy = \frac{1}{\sqrt{1+x^2}}$

Then

$$y' = \frac{1}{1+x^2}$$

If u is a differential function of x Then

$$\frac{dy}{dx} = \frac{1}{1+u^2}\frac{du}{dx}$$

TABLE Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

Ex 3.9 Pages 191. Problems 1,2,3, 9,12, 21-34

2.3. Some integrations lead to inverse trigonometric functions

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$$

Which can be determined from rules 1,3,5 of the derivatives of the inverses

1) $\int \frac{dx}{\sqrt{9-x^2}} = \int \frac{dx}{\sqrt{3^2-x^2}} = sin^{-1}\left(\frac{x}{3}\right) + c$

2) $\int \frac{dx}{4+x^2} = \frac{1}{2}tan^{-1}\left(\frac{x}{2}\right) + C$

3) $\int \frac{dx}{2x\sqrt{(2x)^2-9}} = \frac{1}{3}sec^{-1}\left|\frac{2x}{3}\right| + C$

3. Natural logarithm

Definition of the Natural Logarithm Function

The natural logarithm of a positive number x, written as $\ln x$, is the value of an integral.

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0$$

DEFINITION The **number** e is that number in the domain of the natural logarithm satisfying

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1$$

The Derivative of $y = \ln x$

By the first part of the Fundamental Theorem of Calculus (Section 5.4),

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}.$$

For every positive value of x, we have

$$\frac{d}{dx}\ln x = \frac{1}{x}. \tag{1}$$

Therefore, the function $y = \ln x$ is a solution to the initial value problem $dy/dx = 1/x$, $x > 0$, with $y(1) = 0$. Notice that the derivative is always positive.

If u is a differentiable function of x whose values are positive, so that $\ln u$ is defined, then applying the Chain Rule we obtain

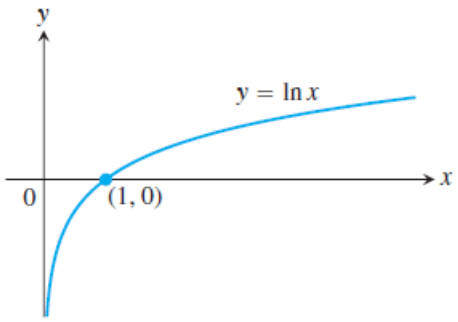
$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \tag{2}$$

3.1 Some natural logarithmic properties

1. $\ln bx = \ln b + \ln x$	**2.** $\ln \frac{b}{x} = \ln b - \ln x$
3. $\ln \frac{1}{x} = -\ln x$	**4.** $\ln x^r = r \ln x$

3.2 The Graph and Range of ln *x*

The derivative $d(\ln x)/dx = 1/x$ is positive for $x > 0$, so $\ln x$ is an increasing function of x. The second derivative, $-1/x^2$, is negative, so the graph of $\ln x$ is concave down.



3.3 The integral of $\int \ln u \, du$

Equation (**2**) leads to the following integral formula.

If u is a differentiable function that is never zero,

$$\int \frac{1}{u} du = \ln |u| + C. \tag{4}$$

EXAMPLE Here we recognize an integral of the form $\int \frac{du}{u}$.

$$\int_{-\pi/2}^{\pi/2} \frac{4 \cos \theta}{3 + 2 \sin \theta} d\theta = \int_1^5 \frac{2}{u} du \qquad u = 3 + 2 \sin \theta, \quad du = 2 \cos \theta \, d\theta, \quad u(-\pi/2) = 1, \quad u(\pi/2) = 5$$

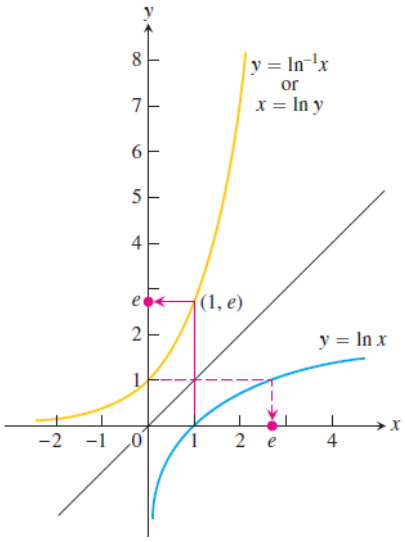
$$= 2 \ln |u| \Big]_1^5$$

$$= 2 \ln |5| - 2 \ln |1| = 2 \ln 5$$

Note that $u = 3 + 2 \sin \theta$ is always positive on $[-\pi/2, \pi/2]$, so Equation (4) applies. ■

3.4 The inverse of ln x and the number e

The function ln x, being an increasing function of x with domain $(0, \infty)$ and range $(-\infty, \infty)$, has an inverse $\ln^{-1} x$ with domain $(-\infty, \infty)$ and range $(0, \infty)$. The graph of $\ln^{-1} x$ is the graph of ln x reflected across the line $y = x$. As you can see in Figure



To find the inverse of y=ln x , Then $x = e^y$ and by substitution $y = e^x$ and by taking ln for both sides then $x = \ln y$ which is shown in the yellow line

3.5 The derivative and integral of e^x

Let $y = e^x$

$y' = e^x \ ln \ e = e^x$ since ln e =1

If u is any differentiable function of x, then

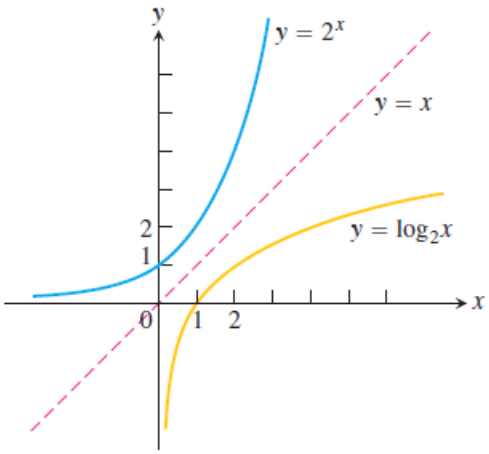
$$\frac{d}{dx} e^u = e^u \frac{du}{dx}.$$

And

$$\int e^u \, du = e^u + C.$$

4.Logarithms with Base *a*

DEFINITION For any positive number $a \neq 1$, the **logarithm of x with base a**, denoted by $\log_a x$, is the inverse function of a^x.



As we know earlier that

$x = a^y$ .....(1)

is equivalent to $\log_a x = y$

which is the inverse of $y = a^x$

also, by taking ln for both side of eq 1 then

$lnx = ylna$

$$y = \frac{lnx}{lna} = \log_a x$$

Some rules for logarithms which can easily be proved

TABLE Rules for base a logarithms

For any numbers $x > 0$ and $y > 0$,

1. *Product Rule:*
$$\log_a xy = \log_a x + \log_a y$$

2. *Quotient Rule:*
$$\log_a \frac{x}{y} = \log_a x - \log_a y$$

3. *Reciprocal Rule:*
$$\log_a \frac{1}{y} = -\log_a y$$

4. *Power Rule:*
$$\log_a x^y = y \log_a x$$

4.1 Derivative and Integrals Involving $\log_a x$

To find derivatives or integrals involving base a logarithms, we convert them to natural logarithms. If u is a positive differentiable function of x, then

$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{\ln u}{\ln a}\right) = \frac{1}{\ln a}\frac{d}{dx}(\ln u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}.$$

$$\boxed{\frac{d}{dx}(\log_a u) = \frac{1}{\ln a}\cdot\frac{1}{u}\frac{du}{dx}}$$

EXAMPLE We illustrate the derivative and integral results.

(a) $$\frac{d}{dx}\log_{10}(3x+1) = \frac{1}{\ln 10}\cdot\frac{1}{3x+1}\frac{d}{dx}(3x+1) = \frac{3}{(\ln 10)(3x+1)}$$

(b) $$\int \frac{\log_2 x}{x}\,dx = \frac{1}{\ln 2}\int \frac{\ln x}{x}\,dx \qquad \log_2 x = \frac{\ln x}{\ln 2}$$

$$= \frac{1}{\ln 2}\int u\,du \qquad u = \ln x, \quad du = \frac{1}{x}\,dx$$

$$= \frac{1}{\ln 2}\frac{u^2}{2} + C = \frac{1}{\ln 2}\frac{(\ln x)^2}{2} + C = \frac{(\ln x)^2}{2\ln 2} + C$$ ■

Ex 7.1 Pages 425, problem 1-46