## Complex number

A Complex Number is a combination of a **Real Number** and an **Imaginary Number**




**Where**

$$i = \sqrt{-1}$$

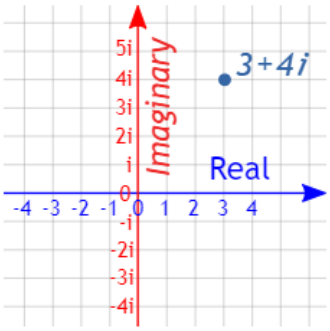
Examples:

| | | | | |
|---|---|---|---|---|
| 1 + i | 39 + 3i | 0.8 − 2.2i | −2 + πi | √2 + i/2 |

So, a Complex Number has a real part and an imaginary part. But either part can be **0**.

| Complex Number | Real Part | Imaginary Part | |
|---|---|---|---|
| 3 + 2i | 3 | 2 | |
| 5 | 5 | **0** | Purely Real |
| −6i | **0** | −6 | Purely Imaginary |

A complex number can now be shown as a point:




The complex number 3 + 4**i**

## Adding

To add two complex numbers, we add each part separately:

$$(a+b\mathbf{i}) + (c+d\mathbf{i}) = (a+c) + (b+d)\mathbf{i}$$

Example: add the complex numbers $\mathbf{3 + 2}\boldsymbol{i}$ and $\mathbf{1 + 7}\boldsymbol{i}$

$$(3 + 2i) + (1 + 7i)$$
$$= 3 + 1 + (2 + 7)i = 4 + 9i$$

## Multiplying

To multiply complex numbers: **Each part of the first complex number** gets multiplied by **each part of the second complex number**

Example: $(3 + 2i)(1 + 7i)$

$$\begin{aligned}(3 + 2i)(1 + 7i) &= 3\times1 + 3\times7i + 2i\times1 + 2i\times7i \\ &= 3 + 21i + 2i + 14i^2 \\ &= 3 + 21i + 2i - 14 \qquad (\text{because } i^2 = -1) \\ &= -11 + 23i\end{aligned}$$

And this:

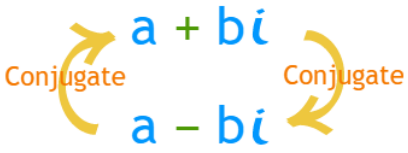
Example: $(1 + i)^2$

$$\begin{aligned}(1 + i)(1 + i) &= 1\times1 + 1\times i + 1\times i + i^2 \\ &= 1 + 2i - 1 \qquad (\text{because } i^2 = -1) \\ &= 0 + 2i\end{aligned}$$

# Conjugates

We will need to know about conjugates in a minute!

A conjugate is where we **change the sign in the middle** like this:




A conjugate is often written with a bar over it:

**Example:**

$$\overline{5 - 3\mathbf{i}} = 5 + 3\mathbf{i}$$

# Dividing

The conjugate is used to help complex division.

- **multiply both top and bottom** by the **conjugate of the bottom**.

**Example: Do this Division:**

$$\frac{2 + 3\mathbf{i}}{4 - 5\mathbf{i}}$$

Multiply top and bottom by the conjugate of $4 - 5\mathbf{i}$ :

$$\frac{2 + 3\mathbf{i}}{4 - 5\mathbf{i}} \times \frac{4 + 5\mathbf{i}}{4 + 5\mathbf{i}} = \frac{8 + 10\mathbf{i} + 12\mathbf{i} + 15\mathbf{i}^2}{16 + 20\mathbf{i} - 20\mathbf{i} - 25\mathbf{i}^2}$$

Now remember that $\mathbf{i}^2 = -1$, so:

$$= \frac{8 + 10\mathbf{i} + 12\mathbf{i} - 15}{16 + 20\mathbf{i} - 20\mathbf{i} + 25}$$

Add Like Terms (and notice how on the bottom $20\mathbf{i} - 20\mathbf{i}$ cancels out!):

$$= \frac{-7 + 22\mathbf{i}}{41}$$

Lastly we should put the answer back into $a + b\mathbf{i}$ form:

$$= \frac{-7}{41} + \frac{22}{41}\mathbf{i}$$

## Multiplying by the Conjugate

The general rule is:

$$(a + b\mathbf{i})(a - b\mathbf{i}) = a^2 + b^2$$

**Example**

**Multiply the complex number 4+5i by its conjugate**

$$(4 - 5\mathbf{i})(4 + 5\mathbf{i}) = 16 + 20\mathbf{i} - 20\mathbf{i} - 25\mathbf{i}^2$$

The middle terms ($20\mathbf{i} - 20\mathbf{i}$) cancel out:

$$(4 - 5\mathbf{i})(4 + 5\mathbf{i}) = 16 - 25\mathbf{i}^2$$

Also $\mathbf{i}^2 = -1$ :

$$(4 - 5\mathbf{i})(4 + 5\mathbf{i}) = 16 + 25$$

And 16 and 25 are (magically) squares of the 4 and 5:

$$(4 - 5\mathbf{i})(4 + 5\mathbf{i}) = 4^2 + 5^2$$

$$=41$$

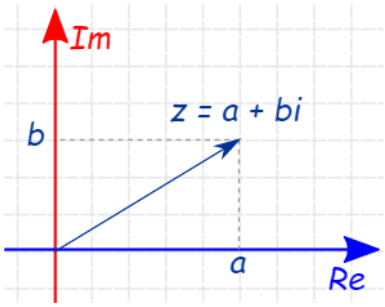
## Notation

We often use **z** for a complex number. And **Re()** for the real part and **Im()** for the imaginary part, like this:

$$z = a + bi$$

Re(z) = a  Im(z) = b

Which looks like this on the complex plane:




## Modulus

Let z = a+ib be a complex number. The Modulus of z is represented by |z|.

Mathematically,

$$|Z| = \sqrt{a^2 + b^2}$$

### Argand Diagrams

There are two geometric representations of the complex number $z = x + iy$:

1. as the point $P(x, y)$ in the $xy$-plane
2. as the vector $\overrightarrow{OP}$ from the origin to $P$.

The x-axis is called the real axis and the y-axis is the imaginary axis both represent the argand diagram for x+iy.

In polar coordinate x=r cos $\theta$ , y=rsin $\theta$ and z=x+iy=r(cos $\theta$ +i sin $\theta$) and

$$|z| = |x + iy| = \sqrt{x^2 + y^2}$$

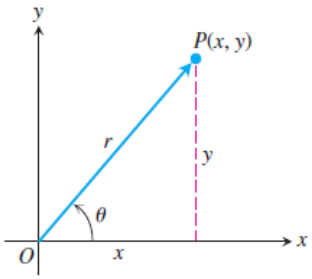



**FIGURE** This Argand diagram represents $z = x + iy$ both as a point $P(x, y)$ and as a vector $\overrightarrow{OP}$.

## Polar Form of Complex Numbers. Powers and Roots

We gain further insight into the arithmetic operations of complex numbers if, in addition to the $xy$-coordinates in the complex plane, we also employ the usual polar coordinates $r, \theta$ defined by

**(1)** $$x = r\cos\theta, \qquad y = r\sin\theta.$$

We see that then $z = x + iy$ takes the so-called **polar form**

**(2)** $$z = r(\cos\theta + i\sin\theta).$$

$r$ is called the **absolute value** or **modulus** of $z$ and is denoted by $|z|$. Hence

**(3)** $$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}.$$

Geometrically, $|z|$ is the distance of the point $z$ from the origin (Fig. 323). Similarly, $|z_1 - z_2|$ is the distance between $z_1$ and $z_2$ (Fig. 324).

$\theta$ is called the **argument** of $z$ and is denoted by $\arg z$. Thus $\theta = \arg z$, see fig below

**(4)** $$\tan\theta = \frac{y}{x} \qquad (z \neq 0).$$

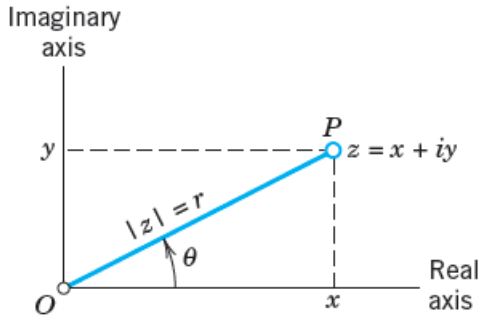



**Fig.** Complex plane, polar form of a complex number

For polar coordinate, x=cos$\theta$ and y=r sin$\theta$ then

$$z = x + iy = r(cos\theta + isin\theta)$$

But

$$e^{ix} = \cos x + i\, sinx$$

Then for r=1

$$e^{i\theta} = \cos\theta + i\, sin\theta$$ which is called Euler's Formula

## Euler's Formula

The identity

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

called **Euler's formula**, enables us to rewrite Equation ( 2 ) as

$$z = re^{i\theta}.$$

This formula, in turn, leads to the following rules for calculating products, quotients, powers, and roots of complex numbers. It also leads to Argand diagrams for $e^{i\theta}$. Since $\cos\theta + i\sin\theta$ is what we get from Equation ( 2 ) by taking $r = 1$, we can say that $e^{i\theta}$ is represented by a unit vector that makes an angle $\theta$ with the positive $x$-axis, as shown .

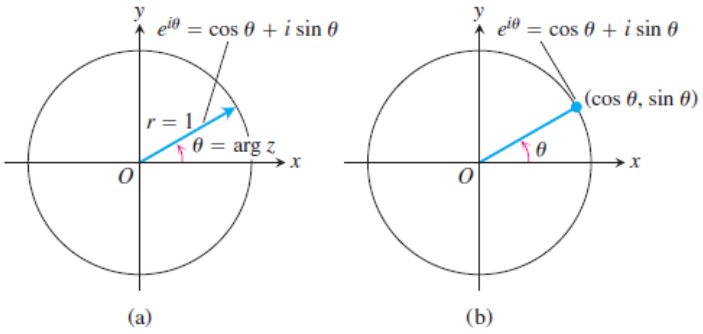



**FIGURE** Argand diagrams for $e^{i\theta} = \cos\theta + i\sin\theta$ (a) as a vector and (b) as a point.

## Products

To multiply two complex numbers, we multiply their absolute values and add their angles. Let

$$z_1 = r_1 e^{i\theta_1}, \qquad z_2 = r_2 e^{i\theta_2},$$

so that

$$|z_1| = r_1, \qquad \arg z_1 = \theta_1; \qquad |z_2| = r_2, \qquad \arg z_2 = \theta_2.$$

Then

$$z_1 z_2 = r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

and hence

$$|z_1 z_2| = r_1 r_2 = |z_1| \cdot |z_2|$$
$$\arg(z_1 z_2) = \theta_1 + \theta_2 = \arg z_1 + \arg z_2.$$

**EXAMPLE 2** Let $z_1 = 1 + i, z_2 = \sqrt{3} - i$. We plot these complex numbers in an Argand diagram (Figure A.27) from which we read off the polar representations

$$z_1 = \sqrt{2}e^{i\pi/4}, \qquad z_2 = 2e^{-i\pi/6}.$$

Then

$$z_1 z_2 = 2\sqrt{2}\exp\left(\frac{i\pi}{4} - \frac{i\pi}{6}\right) = 2\sqrt{2}\exp\left(\frac{i\pi}{12}\right)$$
$$= 2\sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right) \approx 2.73 + 0.73i.$$

### Quotients

Suppose $r_2 \neq 0$ then

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1-\theta_2)}.$$

Hence

$$\left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|} \quad \text{and} \quad \arg\left(\frac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2.$$

That is, we divide lengths and subtract angles for the quotient of complex numbers.

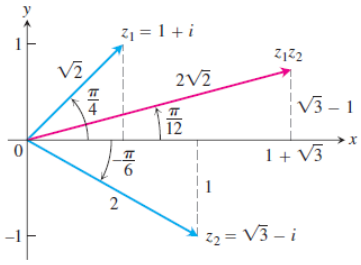



FIGURE A.27 To multiply two complex numbers, multiply their absolute values

**EXAMPLE** Let $z_1 = 1 + i$ and $z_2 = \sqrt{3} - i$, as in Example 2. Then

$$\frac{1+i}{\sqrt{3}-i} = \frac{\sqrt{2}e^{i\pi/4}}{2e^{-i\pi/6}} = \frac{\sqrt{2}}{2}e^{5\pi i/12} \approx 0.707\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$$
$$\approx 0.183 + 0.683i.$$

## Powers

If $n$ is a positive integer, we may apply the product formulas in Equation (12) to find

$$z^n = z \cdot z \cdot \cdots \cdot z. \qquad n \text{ factors}$$

With $z = re^{i\theta}$, we obtain

$$\begin{aligned} z^n = (re^{i\theta})^n &= r^n e^{i(\theta+\theta+\cdots+\theta)} \qquad n \text{ summands} \\ &= r^n e^{in\theta}. \end{aligned} \tag{13}$$

The length $r = |z|$ is raised to the $n$th power and the angle $\theta = \arg z$ is multiplied by $n$.

If we take $r = 1$ in Equation (13), we obtain De Moivre's Theorem.

### De Moivre's Theorem

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta. \tag{14}$$

If we expand the left side of De Moivre's equation above by the Binomial Theorem and reduce it to the form $a + ib$, we obtain formulas for $\cos n\theta$ and $\sin n\theta$ as polynomials of degree $n$ in $\cos\theta$ and $\sin\theta$.

**EXAMPLE 4** If $n = 3$ in Equation (14), we have

$$(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta.$$

The left side of this equation expands to

$$\cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

The real part of this must equal $\cos 3\theta$ and the imaginary part must equal $\sin 3\theta$. Therefore,

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta,$$

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

■

## Roots

If $z = re^{i\theta}$ is a complex number different from zero and $n$ is a positive integer, then there are precisely $n$ different complex numbers $w_0, w_1, \ldots, w_{n-1}$, that are $n$th roots of $z$. To see why, let $w = \rho e^{i\alpha}$ be an $n$th root of $z = re^{i\theta}$, so that

$$w^n = z$$

or

$$\rho^n e^{in\alpha} = re^{i\theta}.$$

Then

$$\rho = \sqrt[n]{r}$$

is the real, positive $n$th root of $r$. For the argument, although we cannot say that $n\alpha$ and $\theta$ must be equal, we can say that they may differ only by an integer multiple of $2\pi$. That is,

$$n\alpha = \theta + 2k\pi, \qquad k = 0, \pm 1, \pm 2, \ldots .$$

Therefore,

$$\alpha = \frac{\theta}{n} + k\frac{2\pi}{n}.$$

Hence, all the $n$th roots of $z = re^{i\theta}$ are given by

$$\sqrt[n]{re^{i\theta}} = \sqrt[n]{r}\exp i\left(\frac{\theta}{n} + k\frac{2\pi}{n}\right), \qquad k = 0, \pm 1, \pm 2, .$$

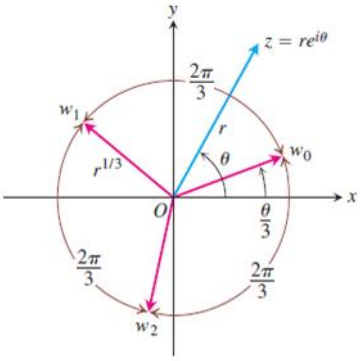



**FIGURE** The three cube roots of $z = re^{i\theta}$.

**EXAMPLE** Find the four fourth roots of $-16$.

**Solution** As our first step, we plot the number $-16$ in an Argand diagram (Figure A.29) and determine its polar representation $re^{i\theta}$. Here, $z = -16$, $r = +16$, and $\theta = \pi$. One of the fourth roots of $16e^{i\pi}$ is $2e^{i\pi/4}$. We obtain others by successive additions of $2\pi/4 = \pi/2$ to the argument of this first one. Hence,

$$\sqrt[4]{16 \exp i\pi} = 2 \exp i\left(\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\right),$$

and the four roots are

$$w_0 = 2\left[\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right] = \sqrt{2}(1 + i)$$

$$w_1 = 2\left[\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right] = \sqrt{2}(-1 + i)$$

$$w_2 = 2\left[\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right] = \sqrt{2}(-1 - i)$$

$$w_3 = 2\left[\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right] = \sqrt{2}(1 - i).$$

■

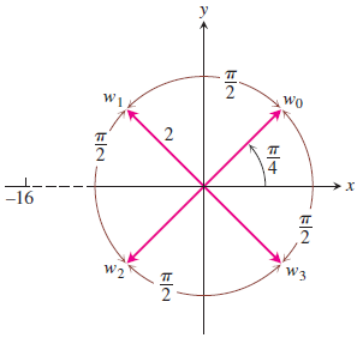
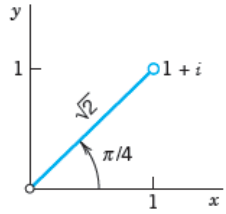



**FIGURE** The four fourth roots of $-16$.

**EXAMPLE** **Polar Form of Complex Numbers. Principal Value Arg z**




$z = 1 + i$ (Fig. ) has the polar form $z = \sqrt{2}\,(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$. Hence we obtain

$$|z| = \sqrt{2}, \quad \arg z = \tfrac{1}{4}\pi \pm 2n\pi \ (n = 0, 1, \cdots), \qquad \text{and} \qquad \operatorname{Arg} z = \tfrac{1}{4}\pi \quad \text{(the principal value)}.$$

Similarly, $z = 3 + 3\sqrt{3}i = 6\,(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$, $|z| = 6$, and $\operatorname{Arg} z = \frac{1}{3}\pi$.

Solve the following problems

## 1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$
2. $-4 + 4i$
3. $2i, \quad -2i$
4. $-5$
5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$
6. $\dfrac{\sqrt{3} - 10i}{-\frac{1}{2}\sqrt{3} + 5i}$
7. $1 + \frac{1}{2}\pi i$
8. $\dfrac{-4 + 19i}{2 + 5i}$

## 9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $-1 + i$
10. $-5, \quad -5 - i, \quad -5 + i$
11. $3 \pm 4i$
12. $-\pi - \pi i$
13. $(1 + i)^{20}$
14. $-1 + 0.1i, \quad -1 - 0.1i$

## 15–18 CONVERSION TO $x + iy$

Graph in the complex plane and represent in the form $x + iy$:

15. $3\left(\cos \frac{1}{2}\pi - i \sin \frac{1}{2}\pi\right)$
16. $6\left(\cos \frac{1}{3}\pi + i \sin \frac{1}{3}\pi\right)$
17. $\sqrt{8}\left(\cos \frac{1}{4}\pi + i \sin \frac{1}{4}\pi\right)$
18. $\sqrt{50}\left(\cos \frac{3}{4}\pi + i \sin \frac{3}{4}\pi\right)$

## 21–27 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt[3]{1 + i}$
22. $\sqrt[3]{3 + 4i}$
23. $\sqrt[3]{216}$
24. $\sqrt[4]{-4}$
25. $\sqrt[4]{i}$
26. $\sqrt[8]{1}$
27. $\sqrt[5]{-1}$

Solution for odd problems

**1.** $\sqrt{2}\,(\cos\frac{1}{4}\pi + i\sin\frac{1}{4}\pi)$

**3.** $2(\cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi)$, $2(\cos\frac{1}{2}\pi - i\sin\frac{1}{2}\pi)$

---



**5.** $\frac{1}{2}(\cos\pi + i\sin\pi)$

**7.** $\sqrt{1+\frac{1}{4}\pi^2}\,(\cos\arctan\frac{1}{2}\pi + i\sin\arctan\frac{1}{2}\pi)$

**9.** $3\pi/4$

**11.** $\pm\arctan(\frac{4}{3}) = \pm 0.9273$

**13.** $-1024$. *Answer:* $\pi$

**15.** $-3i$

**17.** $2 + 2i$

**21.** $\sqrt[6]{2}\,(\cos\frac{1}{12}k\pi + i\sin\frac{1}{12}\pi)$, $k = 1, 9, 17$

**23.** $6$, $-3 \pm 3\sqrt{3}i$

**25.** $\cos(\frac{1}{8}\pi + \frac{1}{2}k\pi) + i\sin(\frac{1}{8}\pi + \frac{1}{2}k\pi)$, $k = 0, 1, 2, 3$

Solve the following problem

EX.A.7 Page Ap-34

2,10,13,15, 19,24