1.Sigma Notation and Limits of Finite Sums

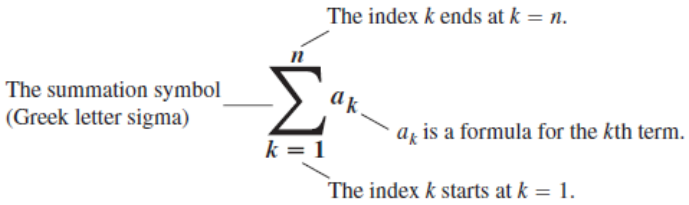
In this section we introduce a convenient notation for sums with a number of terms.

Finite Sums and Sigma Notation

Sigma notation enables us to write a sum with many terms in the compact form

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n.$$

The Greek letter Σ (capital sigma, corresponding to our letter S), stands for "sum." The **index of summation** k tells us where the sum begins (at the number below the Σ symbol) and where it ends (at the number above Σ). Any letter can be used to denote the index, but the letters i, j, and k are customary.



Then we can write

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 + 11^2 = \sum_{k=1}^{11} k^2,$$

and

$$f(1) + f(2) + f(3) + \cdots + f(100) = \sum_{i=1}^{100} f(i).$$

The lower limit of summation does not have to be 1; it can be any integer.

EXAMPLE 1

A sum in sigma notation	The sum written out, one term for each value of k	The value of the sum
$\sum_{k=1}^{5} k$	$1 + 2 + 3 + 4 + 5$	15
$\sum_{k=1}^{3} (-1)^k k$	$(-1)^1(1) + (-1)^2(2) + (-1)^3(3)$	$-1 + 2 - 3 = -2$
$\sum_{k=1}^{2} \frac{k}{k+1}$	$\frac{1}{1+1} + \frac{2}{2+1}$	$\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$
$\sum_{k=4}^{5} \frac{k^2}{k-1}$	$\frac{4^2}{4-1} + \frac{5^2}{5-1}$	$\frac{16}{3} + \frac{25}{4} = \frac{139}{12}$

2.Integration

In this part we study a method to calculate the areas and volumes of very general shapes. This method, called *integration* which is a tool for calculating much more than areas and volumes. The *integral* is of fundamental importance in statistics, the sciences, and engineering. We use it to calculate quantities ranging from probabilities and averages to energy consumption and the forces against a dam's floodgates. We focus on the integral concept and its use in computing areas of various regions with curved boundaries.

The Fundamental Theorem tells us how to evaluate a definite integral once we have an antiderivative for the integrand function. Table 8.1 summarizes the forms of antiderivatives for many of the functions we have studied so far, and the substitution method helps us use the table to evaluate more complicated functions involving these basic ones. In this chapter we study a number of other important techniques for finding antiderivatives (or indefinite integrals) for many combinations of functions whose antiderivatives cannot be found using the methods presented before

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C$ (any number k)
2. $\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1)$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int e^x\,dx = e^x + C$
5. $\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\int \sin x\,dx = -\cos x + C$
7. $\int \cos x\,dx = \sin x + C$
8. $\int \sec^2 x\,dx = \tan x + C$
9. $\int \csc^2 x\,dx = -\cot x + C$
10. $\int \sec x \tan x\,dx = \sec x + C$
11. $\int \csc x \cot x\,dx = -\csc x + C$
12. $\int \tan x\,dx = \ln|\sec x| + C$
13. $\int \cot x\,dx = \ln|\sin x| + C$
14. $\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\int \sinh x\,dx = \cosh x + C$
17. $\int \cosh x\,dx = \sinh x + C$
18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

2.2.Evaluate the following integrals

1) $\int 3x^2\,dx$

2) $\int\left(\frac{1}{x^2}+x\right)dx$

3) $\int x\sqrt{x^2+1}\,dx$

4) $\int\left(2t+t^{-1}\right)^2\,dt$

5) $\int\sqrt{(z^2-z^{-2})^2+4}\,dz$

6) $\int\frac{x+3}{\sqrt{x^2+6x}}\,dx$

7) $\int\frac{x+2}{x^2}\,dx$

8) $\int\frac{e^x}{1+3e^x}\,dx$

9) $\int 3x^3\cdot e^{-2x^4}\,dx$

10) $\int 2^{-4x}\,dx$

Sol. –

1) $\int 3x^2\,dx=3\int x^2dx=3\frac{x^3}{3}+c=x^3+c$

2) $\left(x^{-2}+x\right)dx=\int x^{-2}\,dx+\int x\,dx=\frac{x^{-1}}{-1}+\frac{x^2}{2}+c=-\frac{1}{x}+\frac{x^2}{2}+c$

3) $\int x\sqrt{x^2+1}\,dx=\frac{1}{2}\int 2x(x^2+1)^{1/2}dx=\frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2}+c=\frac{1}{3}\sqrt{(x^2+1)3}+c$

4) $\int\left(2t+t^{-1}\right)^2\,dt=\int\left(4t^2+4+t^{-2}\right)dt=4\frac{t^3}{3}+4t+\frac{t^{-1}}{-1}+c=\frac{4}{3}t^3+4t-\frac{1}{t}+c$

5) $\int\sqrt{(z^2-z^{-2})^2+4}\,dz=\int\sqrt{z^4-2+z^{-4}+4}\,dz=\int\sqrt{z^4+2+z^{-4}}\,dz$

$=\int\sqrt{(z^2+z^{-2})^2}\,dz=\int(z^2+z^{-2})\,dz=\frac{z^3}{3}+\frac{z^{-1}}{-1}+c=\frac{1}{3}z^3-\frac{1}{z}+c$

6) $\int\frac{x+3}{\sqrt{x^2+6x}}\,dx=\frac{1}{2}\int(2x+6)\cdot(x^2+6x)^{-1/2}dx$

$=\frac{1}{2}\cdot\frac{(x^2+6x)^{1/2}}{1/2}+c=\sqrt{x^2+6x}+c$

7) $\int\frac{x+2}{x^2}\,dx=\int\left(\frac{x}{x^2}+\frac{2}{x^2}\right)dx=\int\left(x^{-1}+2x^{-2}\right)dx=\ln x+\frac{2x^{-1}}{-1}+c=\ln x-\frac{2}{x}+c$

8) $\int\frac{e^x}{1+3e^x}\,dx=\frac{1}{3}\int 3e^x(1+3e^x)^{-1}\,dx=\frac{1}{3}\ln(1+3e^x)+c$

9) $\int 3x^3\cdot e^{-2x^4}\,dx=-\frac{3}{8}\int-8x^3\cdot e^{-2x^4}\,dx=-\frac{3}{8}\cdot e^{-2x^4}+c$

10) $\int 2^{-4x}\,dx=-\frac{1}{4}\int 2^{-4x}\cdot(-4dx)=-\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2}+c$

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2.3. Trigonometric Integrals

Some of the following trigonometry identities may be needed.

- A.) $\cos^2 x + \sin^2 x = 1$
- B.) $\sin 2x = 2 \sin x \cos x$
- C.) $\cos 2x = 2\cos^2 x - 1$ so that $\cos^2 x = \dfrac{1 + \cos 2x}{2}$
- D.) $\cos 2x = 1 - 2\sin^2 x$ so that $\sin^2 x = \dfrac{1 - \cos 2x}{2}$
- E.) $\cos 2x = \cos^2 x - \sin^2 x$
- F.) $1 + \tan^2 x = \sec^2 x$ so that $\tan^2 x = \sec^2 x - 1$
- G.) $1 + \cot^2 x = \csc^2 x$ so that $\cot^2 x = \csc^2 x - 1$

2.4 The direct integration for trig. Functions

- 1.) $\int \cos x \, dx = \sin x + C$
- 2.) $\int \sin x \, dx = -\cos x + C$
- 3.) $\int \sec^2 x \, dx = \tan x + C$
- 4.) $\int \csc^2 x \, dx = -\cot x + C$
- 5.) $\int \sec x \tan x \, dx = \sec x + C$
- 6.) $\int \csc x \cot x \, dx = -\csc x + C$
- 7.) $\int \tan x \, dx = \ln|\sec x| + C$
- 8.) $\int \cot x \, dx = \ln|\sin x| + C$
- 9.) $\int \sec x \, dx = \ln|\sec x + \tan x| + C$
- 10.) $\int \csc x \, dx = \ln|\csc x - \cot x| + C$

EX-2- Evaluate the following integrals:

1) $\int \cos(3\theta - 1)\,d\theta$

2) $\int x \cdot \sin(2x^2)\,dx$

3) $\int \cos^2(2y) \cdot \sin(2y)\,dy$

4) $\int \sec^3 x \cdot \tan x\ dx$

5) $\int \sqrt{2 + \sin 3t} \cdot \cos 3t\ dt$

6) $\int \dfrac{d\theta}{\cos^2 \theta}$

7) $\int (1 - \sin^2 3t) \cdot \cos 3t\ dt$

8) $\int \tan^3(5x) \cdot \sec^2(5x)\ dx$

9) $\int \sin^4 x \cdot \cos^3 x\ dx$

10) $\int \dfrac{\cot^2 \sqrt{x}}{\sqrt{x}}\,dx$

Sol.-

1) $\dfrac{1}{3}\int 3\cos(3\theta - 1)\,d\theta = \dfrac{1}{3}\sin(3\theta - 1) + c$

2) $\dfrac{1}{4}\int 4x \cdot \sin(2x^2)\,dx = -\dfrac{1}{4}\cos(2x^2) + c$

3) $-\dfrac{1}{2}\int (\cos 2y)^2 \cdot (-2\sin 2y\ dy) = -\dfrac{1}{2} \cdot \dfrac{(\cos 2y)^3}{3} + c = -\dfrac{1}{6}(\cos 2y)^3 + c$

4) $\int \sec^2 x \cdot (\sec x \cdot \tan x \cdot dx) = \dfrac{\sec^3 x}{3} + c$

5) $\dfrac{1}{3}\int (2 + \sin 3t)^{1/2}(3\cos 3t\ dt) = \dfrac{1}{3} \cdot \dfrac{(2 + \sin 3t)^{3/2}}{3/2} + c = \dfrac{2}{9}\sqrt{(2 + \sin 3t)^3} + c$

6) $\int \dfrac{d\theta}{\cos^2 \theta} = \int \sec^2 \theta \cdot d\theta = \tan \theta + c$

7) $\int (1 - \sin^2 3t) \cdot \cos 3t\ dt = \dfrac{1}{3}\int 3\cos 3t\ dt - \dfrac{1}{3}\int (\sin 3t)^2 \cdot 3\cos 3t\ dt$

$= \dfrac{1}{3}\sin 3t - \dfrac{1}{3} \cdot \dfrac{\sin^3 3t}{3} + c = \dfrac{1}{3} \cdot \sin 3t - \dfrac{1}{9}\sin^3 3t + c$

8) $\dfrac{1}{5}\int \tan^3 5x \cdot (5\sec^2 5x\ dx) = \dfrac{1}{5} \cdot \dfrac{\tan^4 5x}{4} + c = \dfrac{1}{20}\tan^4 5x + c$

9) $\int \sin^4 x \cdot \cos^3 x\ dx = \int \sin^4 x \cdot (1 - \sin^2 x) \cdot \cos x\ dx$

$= \int \sin^4 x \cdot \cos x\ dx - \int \sin^6 x \cdot \cos x\ dx = \dfrac{\sin^5 x}{5} - \dfrac{\sin^7 x}{7} + c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}\,dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}\,dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

$$11)\ \int \cos^3 2x\,dx = \int (1-\sin^2 2x)\cos 2x\,dx = \frac{1}{2}\int (1-u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right) + C$$

Where u=sin2x from which du=2cos2xdx

Solve Ex 5.5 Page 342 From 7 to 15

3. Indefinite Integrals and the Substitution Method

It is known that for a function $f(x)$ that has antiderivative $F(x)$ then

$$\int f(x)\,dx = F(x) + C,$$

Where C is an arbitrary constant

3.1 Substitution

You can transform x by a function u that has its derivative du in the integration so that

$$\int u^n\,du = \frac{u^{n+1}}{n+1} + C,$$

EXAMPLE 1 Find the integral $\int (x^3 + x)^5(3x^2 + 1)\,dx$.

Solution We set $u = x^3 + x$. Then

$$du = \frac{du}{dx}\,dx = (3x^2 + 1)\,dx,$$

so that by substitution we have

$$\int (x^3 + x)^5(3x^2 + 1)\,dx = \int u^5\,du \qquad \text{Let } u = x^3 + x,\ du = (3x^2 + 1)\,dx.$$

$$= \frac{u^6}{6} + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{(x^3 + x)^6}{6} + C \qquad \text{Substitute } x^3 + x \text{ for } u. \quad \blacksquare$$

EXAMPLE 2 Find $\int \sqrt{2x + 1}\, dx$.

Solution The integral does not fit the formula

$$\int u^n\, du,$$

with $u = 2x + 1$ and $n = 1/2$, because

$$du = \frac{du}{dx}\, dx = 2\, dx$$

is not precisely dx. The constant factor 2 is missing from the integral. However, we can introduce this factor after the integral sign if we compensate for it by a factor of $1/2$ in front of the integral sign. So we write

$$\begin{aligned}
\int \sqrt{2x + 1}\, dx &= \frac{1}{2}\int \sqrt{\underbrace{2x + 1}_{u}} \cdot \underbrace{2\, dx}_{du} \\
&= \frac{1}{2}\int u^{1/2}\, du && \text{Let } u = 2x + 1,\ du = 2\, dx. \\
&= \frac{1}{2}\,\frac{u^{3/2}}{3/2} + C && \text{Integrate with respect to } u. \\
&= \frac{1}{3}(2x + 1)^{3/2} + C && \text{Substitute } 2x + 1 \text{ for } u.
\end{aligned}$$

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EXAMPLE 3 Find $\int \sec^2(5t + 1) \cdot 5\, dt$.

Solution We substitute $u = 5t + 1$ and $du = 5\, dt$. Then,

$$\begin{aligned}
\int \sec^2(5t + 1) \cdot 5\, dt &= \int \sec^2 u\, du && \text{Let } u = 5t + 1,\ du = 5\, dt. \\
&= \tan u + C && \frac{d}{du}\tan u = \sec^2 u \\
&= \tan(5t + 1) + C && \text{Substitute } 5t + 1 \text{ for } u.
\end{aligned}$$

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EXAMPLE 4 Find $\int \cos(7\theta + 3)\, d\theta$.

Solution We let $u = 7\theta + 3$ so that $du = 7\,d\theta$. The constant factor 7 is missing from the $d\theta$ term in the integral. We can compensate for it by multiplying and dividing by 7, using the same procedure as in Example 2. Then,

$$\int \cos(7\theta + 3)\,d\theta = \frac{1}{7}\int \cos(7\theta + 3)\cdot 7\,d\theta \qquad \text{Place factor } 1/7 \text{ in front of integral.}$$

$$= \frac{1}{7}\int \cos u\,du \qquad \text{Let } u = 7\theta + 3,\ du = 7\,d\theta.$$

$$= \frac{1}{7}\sin u + C \qquad \text{Integrate.}$$

$$= \frac{1}{7}\sin(7\theta + 3) + C \qquad \text{Substitute } 7\theta + 3 \text{ for } u.$$

EXAMPLE 5 Sometimes we observe that a power of x appears in the integrand that is one less than the power of x appearing in the argument of a function we want to integrate. This observation immediately suggests we try a substitution for the higher power of x. This situation occurs in the following integration.

$$\int x^2 e^{x^3}\,dx = \int e^{x^3}\cdot x^2\,dx$$

$$= \int e^u \cdot \frac{1}{3}\,du \qquad \text{Let } u = x^3,\ du = 3x^2\,dx,\ (1/3)\,du = x^2\,dx.$$

$$= \frac{1}{3}\int e^u\,du$$

$$= \frac{1}{3}e^u + C \qquad \text{Integrate with respect to } u.$$

$$= \frac{1}{3}e^{x^3} + C \qquad \text{Replace } u \text{ by } x^3.$$

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EXAMPLE 6 An integrand may require some algebraic manipulation before the substitution method can be applied. This example gives two integrals obtained by multiplying the integrand by an algebraic form equal to 1, leading to an appropriate substitution.

(a) $$\int \frac{dx}{e^x + e^{-x}} = \int \frac{e^x\,dx}{e^{2x} + 1} \qquad \text{Multiply by } (e^x/e^x) = 1.$$

$$= \int \frac{du}{u^2 + 1} \qquad \text{Let } u = e^x,\ u^2 = e^{2x},\ du = e^x\,dx.$$

$$= \tan^{-1}u + C \qquad \text{Integrate with respect to } u.$$

$$= \tan^{-1}(e^x) + C \qquad \text{Replace } u \text{ by } e^x.$$

EXAMPLE 7 Evaluate $\int x\sqrt{2x+1}\,dx$.

Solution Our previous integration in Example 2 suggests the substitution $u = 2x + 1$ with $du = 2\,dx$. Then,

$$\sqrt{2x+1}\,dx = \frac{1}{2}\sqrt{u}\,du.$$

However in this case the integrand contains an extra factor of x multiplying the term $\sqrt{2x+1}$. To adjust for this, we solve the substitution equation $u = 2x + 1$ to obtain $x = (u-1)/2$, and find that

$$x\sqrt{2x+1}\,dx = \frac{1}{2}(u-1)\cdot\frac{1}{2}\sqrt{u}\,du.$$

The integration now becomes

$$\begin{aligned}
\int x\sqrt{2x+1}\,dx &= \frac{1}{4}\int (u-1)\sqrt{u}\,du = \frac{1}{4}\int (u-1)u^{1/2}\,du && \text{Substitute.}\\
&= \frac{1}{4}\int (u^{3/2} - u^{1/2})\,du && \text{Multiply terms.}\\
&= \frac{1}{4}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C && \text{Integrate.}\\
&= \frac{1}{10}(2x+1)^{5/2} - \frac{1}{6}(2x+1)^{3/2} + C && \text{Replace } u \text{ by } 2x+1. \ \blacksquare
\end{aligned}$$

3.2 Integration of sin^2x , cos^2x

EXAMPLE 9

(a) $$\begin{aligned}
\int \sin^2 x\,dx &= \int \frac{1-\cos 2x}{2}\,dx && \sin^2 x = \frac{1-\cos 2x}{2}\\
&= \frac{1}{2}\int (1-\cos 2x)\,dx\\
&= \frac{1}{2}x - \frac{1}{2}\frac{\sin 2x}{2} + C = \frac{x}{2} - \frac{\sin 2x}{4} + C
\end{aligned}$$

(b) $$\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C \qquad \cos^2 x = \frac{1+\cos 2x}{2} \ \blacksquare$$

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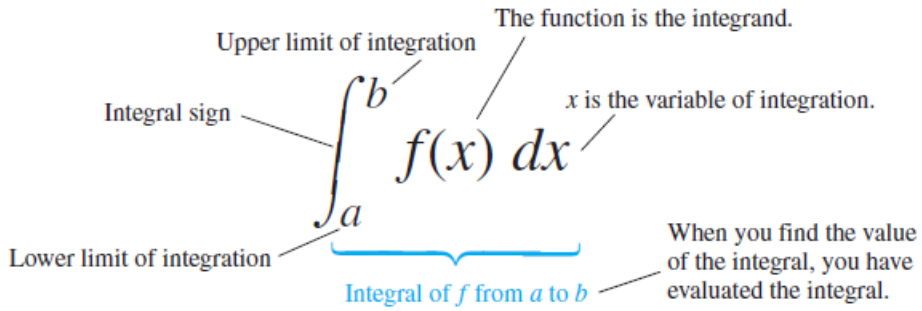
Problem 1-66

4. Definite integral and its rule

The symbol for the definite integral is

$$\int_a^b f(x)\,dx,$$

which is read as "the integral from a to b of f of x dee x" or sometimes as "the integral from a to b of f of x with respect to x." The component parts in the integral symbol also have names:



When f and g are integrable over the interval $[a, b]$, the definite integral satisfies the rules

TABLE Rules satisfied by definite integrals

1.	*Order of Integration:*	$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$	A Definition
2.	*Zero Width Interval:*	$\int_a^a f(x)\,dx = 0$	A Definition when $f(a)$ exists
3.	*Constant Multiple:*	$\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$	Any constant k
4.	*Sum and Difference:*	$\int_a^b (f(x) \pm g(x))\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$	
5.	*Additivity:*	$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$	
6.	*Max-Min Inequality:*	If f has maximum value max f and minimum value min f on $[a, b]$, then $\min f \cdot (b - a) \le \int_a^b f(x)\,dx \le \max f \cdot (b - a).$	
7.	*Domination:*	$f(x) \ge g(x)$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx$	
		$f(x) \ge 0$ on $[a, b] \Rightarrow \int_a^b f(x)\,dx \ge 0$	(Special Case)

It really meant that the defined integral represents the area under the curve and also the finite sum of a rectangle with width of the differential variable and height that represent the function to be integrated

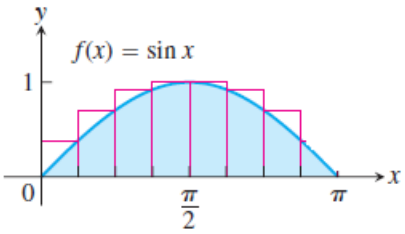
Example

Find the area 1) by integration 2) by summing the rectangles

1) $A = \int_0^{\pi} sinxdx = -cosx]_0^{\pi} = -[-1-1] = 2$

2) $A = \frac{\pi}{8}[sin\frac{\pi}{8} + sin\frac{2\pi}{8} + sin\frac{3\pi}{8} + sin\frac{4\pi}{8} + sin\frac{5\pi}{8} + sin\frac{6\pi}{8} + sin\frac{7\pi}{8} + sin\pi]$

$= \frac{\pi}{8}[0.382 + 0.707 + 0.924 + 1 + 0.924 + 0.707 + 0.382 + 0] = 1.974$



For example the integration of the function sin x from 0 to π is equal to the sum of the area of rectangles as dx approach zero

$A = \int_0^{\pi} sinxdx \approx \sum_{n=1}^{8} sinx.\Delta x$

$A = \lim_{\Delta x \to 0} \sum_{n=1}^{\infty} sinx.\Delta x = \int_0^{\pi} sinxdx$

4.1 Substitution and area between curves

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

EXAMPLE 1 Evaluate $\displaystyle\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$.

Solution We have two choices.

Method 1: Transform the integral and evaluate the transformed integral with the transformed limits given in Theorem 7.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.
When $x = -1$, $u = (-1)^3 + 1 = 0$.
When $x = 1$, $u = (1)^3 + 1 = 2$.

$$= \int_0^2 \sqrt{u}\,du$$

$$= \frac{2}{3}u^{3/2}\Big]_0^2$$

Evaluate the new definite integral.

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

Method 2: Transform the integral as an indefinite integral, integrate, change back to x, and use the original x-limits.

$$\int 3x^2\sqrt{x^3+1}\,dx = \int \sqrt{u}\,du$$

Let $u = x^3 + 1$, $du = 3x^2\,dx$.

$$= \frac{2}{3}u^{3/2} + C$$

Integrate with respect to u.

$$= \frac{2}{3}(x^3+1)^{3/2} + C$$

Replace u by $x^3 + 1$.

$$\int_{-1}^{1} 3x^2\sqrt{x^3+1}\,dx = \frac{2}{3}(x^3+1)^{3/2}\Big]_{-1}^{1}$$

Use the integral just found, with limits of integration for x.

$$= \frac{2}{3}\left[((1)^3+1)^{3/2} - ((-1)^3+1)^{3/2}\right]$$

$$= \frac{2}{3}\left[2^{3/2} - 0^{3/2}\right] = \frac{2}{3}\left[2\sqrt{2}\right] = \frac{4\sqrt{2}}{3}$$

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EXAMPLE 2 We use the method of transforming the limits of integration.

(a) $\displaystyle\int_{\pi/4}^{\pi/2} \cot\theta \csc^2\theta \, d\theta = \int_1^0 u \cdot (-du)$

Let $u = \cot\theta$, $du = -\csc^2\theta \, d\theta$,
$-du = \csc^2\theta \, d\theta$.
When $\theta = \pi/4$, $u = \cot(\pi/4) = 1$.
When $\theta = \pi/2$, $u = \cot(\pi/2) = 0$.

$$= -\int_1^0 u \, du$$

$$= -\left[\frac{u^2}{2}\right]_1^0$$

$$= -\left[\frac{(0)^2}{2} - \frac{(1)^2}{2}\right] = \frac{1}{2}$$

(b) $\displaystyle\int_{-\pi/4}^{\pi/4} \tan x \, dx = \int_{-\pi/4}^{\pi/4} \frac{\sin x}{\cos x} \, dx$

$$= -\int_{\sqrt{2}/2}^{\sqrt{2}/2} \frac{du}{u}$$

Let $u = \cos x$, $du = -\sin x \, dx$.
When $x = -\pi/4$, $u = \sqrt{2}/2$.
When $x = \pi/4$, $u = \sqrt{2}/2$.

$$= -\ln|u| \Big]_{\sqrt{2}/2}^{\sqrt{2}/2} = 0$$

Integrate, zero-width interval ■

EXAMPLE 3 Evaluate $\displaystyle\int_{-2}^{2} (x^4 - 4x^2 + 6) \, dx$.

Solution Since $f(x) = x^4 - 4x^2 + 6$ satisfies $f(-x) = f(x)$, it is even on the symmetric interval $[-2, 2]$, so

$$\int_{-2}^{2} (x^4 - 4x^2 + 6) \, dx = 2\int_0^2 (x^4 - 4x^2 + 6) \, dx$$

$$= 2\left[\frac{x^5}{5} - \frac{4}{3}x^3 + 6x\right]_0^2$$

$$= 2\left(\frac{32}{5} - \frac{32}{3} + 12\right) = \frac{232}{15}.$$

■

4.2 Areas Between Curves

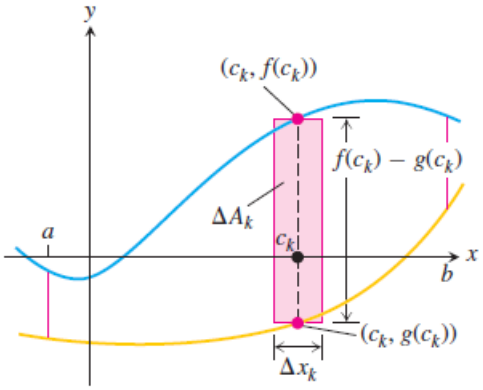


FIGURE The area ΔA_k of the kth rectangle is the product of its height, $f(c_k) - g(c_k)$, and its width, Δx_k.

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

EXAMPLE 4 Find the area of the region bounded above by the curve $y = 2e^{-x} + x$, below by the curve $y = e^x/2$, on the left by $x = 0$, and on the right by $x = 1$.

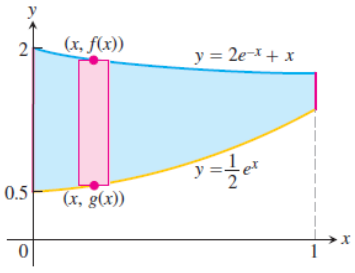


FIGURE 5.28 The region in Example 4

Solution Figure 5.28 displays the graphs of the curves and the region whose area we want to find. The area between the curves over the interval $0 \leq x \leq 1$ is given by

$$A = \int_0^1 \left[(2e^{-x} + x) - \frac{1}{2}e^x\right] dx = \left[-2e^{-x} + \frac{1}{2}x^2 - \frac{1}{2}e^x\right]_0^1$$

$$= \left(-2e^{-1} + \frac{1}{2} - \frac{1}{2}e\right) - \left(-2 + 0 - \frac{1}{2}\right)$$

$$= 3 - \frac{2}{e} - \frac{e}{2} \approx 0.9051. \quad \blacksquare$$

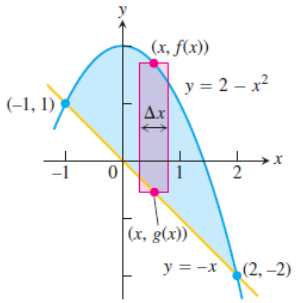


FIGURE 5.29 The region in Example 5 with a typical approximating rectangle.

EXAMPLE 5 Find the area of the region enclosed by the parabola $y = 2 - x^2$ and the line $y = -x$.

Solution First we sketch the two curves (Figure 5.29). The limits of integration are found by solving $y = 2 - x^2$ and $y = -x$ simultaneously for x.

$$\begin{aligned} 2 - x^2 &= -x && \text{Equate } f(x) \text{ and } g(x). \\ x^2 - x - 2 &= 0 && \text{Rewrite.} \\ (x + 1)(x - 2) &= 0 && \text{Factor.} \\ x = -1, \quad & x = 2. && \text{Solve.} \end{aligned}$$

The region runs from $x = -1$ to $x = 2$. The limits of integration are $a = -1$, $b = 2$.

The area between the curves is

$$\begin{aligned} A &= \int_a^b [f(x) - g(x)]\,dx = \int_{-1}^{2} [(2 - x^2) - (-x)]\,dx \\ &= \int_{-1}^{2} (2 + x - x^2)\,dx = \left[2x + \frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{2} \\ &= \left(4 + \frac{4}{2} - \frac{8}{3}\right) - \left(-2 + \frac{1}{2} + \frac{1}{3}\right) = \frac{9}{2} \end{aligned}$$

Richard Dedekind

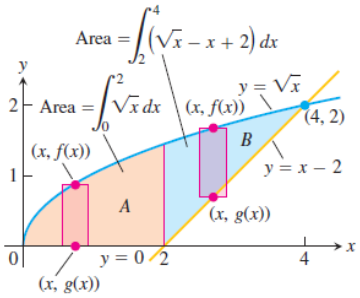


FIGURE 5.30 When the formula for a bounding curve changes, the area integral changes to become the sum of integrals to match, one integral for each of the shaded regions shown here for Example 6.

EXAMPLE 6 Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution The sketch (Figure 5.30) shows that the region's upper boundary is the graph of $f(x) = \sqrt{x}$. The lower boundary changes from $g(x) = 0$ for $0 \le x \le 2$ to $g(x) = x - 2$ for $2 \le x \le 4$ (both formulas agree at $x = 2$). We subdivide the region at $x = 2$ into subregions A and B, shown in Figure 5.30.

The limits of integration for region A are $a = 0$ and $b = 2$. The left-hand limit for region B is $a = 2$. To find the right-hand limit, we solve the equations $y = \sqrt{x}$ and $y = x - 2$ simultaneously for x:

$$\begin{aligned} \sqrt{x} &= x - 2 && \text{Equate } f(x) \text{ and } g(x). \\ x &= (x - 2)^2 = x^2 - 4x + 4 && \text{Square both sides.} \\ x^2 - 5x + 4 &= 0 && \text{Rewrite.} \\ (x - 1)(x - 4) &= 0 && \text{Factor.} \\ x = 1, \quad & x = 4. && \text{Solve.} \end{aligned}$$

Only the value $x = 4$ satisfies the equation $\sqrt{x} = x - 2$. The value $x = 1$ is an extraneous root introduced by squaring. The right-hand limit is $b = 4$.

$$\begin{aligned} &\text{For } 0 \le x \le 2: && f(x) - g(x) = \sqrt{x} - 0 = \sqrt{x} \\ &\text{For } 2 \le x \le 4: && f(x) - g(x) = \sqrt{x} - (x - 2) = \sqrt{x} - x + 2 \end{aligned}$$

We add the areas of subregions A and B to find the total area:

$$\begin{aligned} \text{Total area} &= \underbrace{\int_0^2 \sqrt{x}\,dx}_{\text{area of } A} + \underbrace{\int_2^4 (\sqrt{x} - x + 2)\,dx}_{\text{area of } B} \\ &= \left[\frac{2}{3}x^{3/2}\right]_0^2 + \left[\frac{2}{3}x^{3/2} - \frac{x^2}{2} + 2x\right]_2^4 \\ &= \frac{2}{3}(2)^{3/2} - 0 + \left(\frac{2}{3}(4)^{3/2} - 8 + 8\right) - \left(\frac{2}{3}(2)^{3/2} - 2 + 4\right) \\ &= \frac{2}{3}(8) - 2 = \frac{10}{3}. \end{aligned}$$

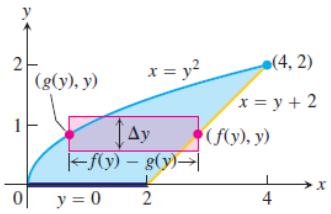


FIGURE 5.31 It takes two integrations to find the area of this region if we integrate with respect to x. It takes only one if we integrate with respect to y (Example 7).

EXAMPLE 7 Find the area of the region in Example 6 by integrating with respect to y.

Solution We first sketch the region and a typical *horizontal* rectangle based on a partition of an interval of y-values (Figure 5.31). The region's right-hand boundary is the line $x = y + 2$, so $f(y) = y + 2$. The left-hand boundary is the curve $x = y^2$, so $g(y) = y^2$. The lower limit of integration is $y = 0$. We find the upper limit by solving $x = y + 2$ and $x = y^2$ simultaneously for y:

$$\begin{aligned} y + 2 &= y^2 && \text{Equate } f(y) = y + 2 \text{ and } g(y) = y^2. \\ y^2 - y - 2 &= 0 && \text{Rewrite.} \\ (y + 1)(y - 2) &= 0 && \text{Factor.} \\ y = -1, \quad y &= 2 && \text{Solve.} \end{aligned}$$

The upper limit of integration is $b = 2$. (The value $y = -1$ gives a point of intersection *below* the x-axis.)

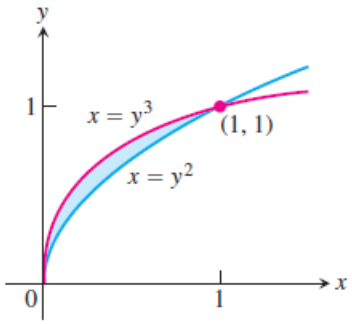
The area of the region is

$$\begin{aligned} A = \int_c^d [f(y) - g(y)]\, dy &= \int_0^2 [y + 2 - y^2]\, dy \\ &= \int_0^2 [2 + y - y^2]\, dy \\ &= \left[2y + \frac{y^2}{2} - \frac{y^3}{3}\right]_0^2 \\ &= 4 + \frac{4}{2} - \frac{8}{3} = \frac{10}{3}. \end{aligned}$$

This is the result of Example 6, found with less work. ■

Example 9

Find the Shaded area in fig below



For perpendicular strip

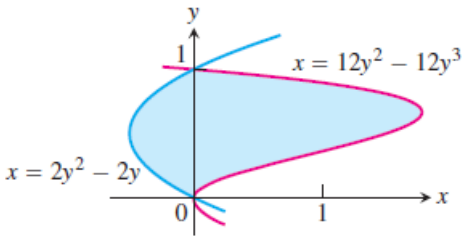
$$A = \int_0^1 \left(\sqrt[3]{x} - \sqrt[2]{x}\right) dx = \frac{3}{4}x^{4/3} - \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{1}{12}$$

For horizontal strip

$$A = \int_0^1 (y^2 - y^3)\, dy = \frac{1}{3}y^3 - \frac{1}{4}y^4\Big]_0^1 = \frac{1}{12}$$

Find the shaded area

The appropriate strip is the horizontal one

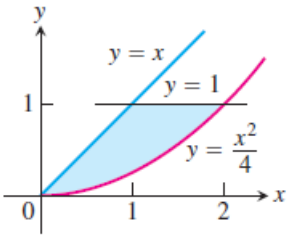


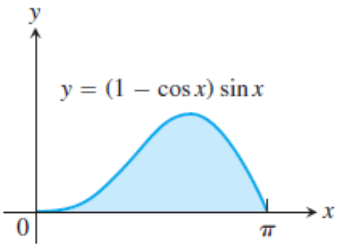
$$A = \int_0^1 (x_{red} - x_{blue})\, dy = \int_0^1 (12y^2 - 12y^3 - 2y^2 + 2y)dy = 4y^3 - 3y^4 - \frac{2}{3}y^3 + y^2]_0^1$$

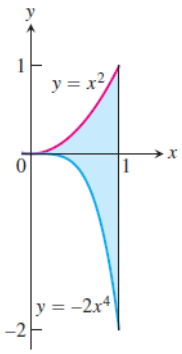
$$= 4 - 3 + \frac{2}{3} + 1 = \frac{8}{3}$$

Home works: For the drawing below find the shaded area

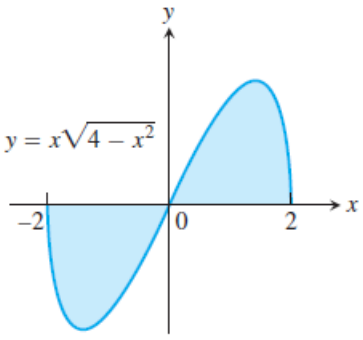
1) **2)** **3)**

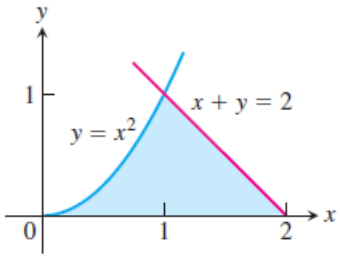


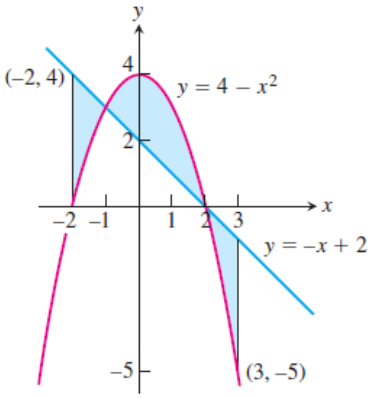




4) **5)** **6)**



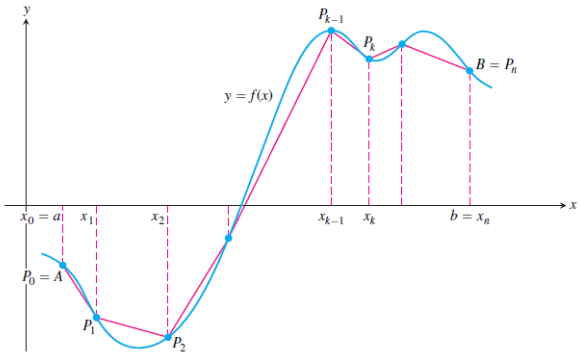




Solve as home works Ex 5.6 Page 350 Problem 1-46 , 63-106

5.Arc length

If the curve is the graph of a continuous function defined over an interval, then we can find the length of the curve using a procedure similar to that we used for defining the area between the curve and the *x*-axis. This procedure results in a division of the curve from point *A* to point *B* into many pieces and joining successive points of division by straight line segments. We then sum the lengths of all these line segments and define the length of the curve to be the limiting value of this sum as the number of segments goes to infinity.



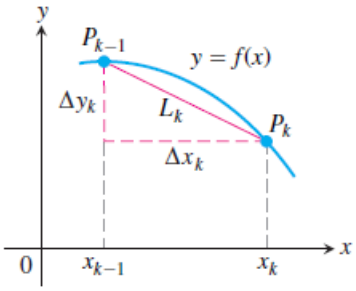


FIGURE The arc $P_{k-1}P_k$ of the curve $y = f(x)$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

Then

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

But

$$\Delta y_k = f'(c_k)\,\Delta x_k.$$

Then the length of the curve between a ,b is

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\,\Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx.$$

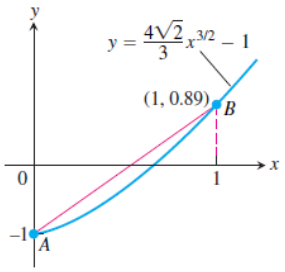


FIGURE The length of the curve is slightly larger than the length of the line segment joining points A and B (Example 1).

EXAMPLE 1 Find the length of the curve (Figure)

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution Use eq. of arc length with $a = 0$, $b = 1$, and

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1 \qquad x = 1, y \approx 0.89$$

$$\frac{dy}{dx} = \frac{4\sqrt{2}}{3} \cdot \frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}$$

$$\left(\frac{dy}{dx}\right)^2 = \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.$$

The length of the curve over $x = 0$ to $x = 1$ is

$$L = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1 + 8x}\,dx \qquad \text{Eq. (3) with } a = 0, b = 1$$

$$= \frac{2}{3} \cdot \frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6} \approx 2.17. \qquad \text{Let } u = 1 + 8x, \text{ integrate, and replace } u \text{ by } 1 + 8x.$$

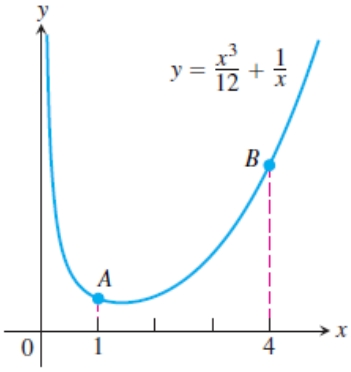


FIGURE The curve in Example 2, where $A = (1, 13/12)$ and $B = (4, 67/12)$.

EXAMPLE Find the length of the graph of

$$f(x) = \frac{x^3}{12} + \frac{1}{x}, \qquad 1 \le x \le 4$$

Solution A graph of the function is shown in Figure

$$f'(x) = \frac{x^2}{4} - \frac{1}{x^2}$$

so

$$1 + [f'(x)]^2 = 1 + \left(\frac{x^2}{4} - \frac{1}{x^2}\right)^2 = 1 + \left(\frac{x^4}{16} - \frac{1}{2} + \frac{1}{x^4}\right)$$

$$= \frac{x^4}{16} + \frac{1}{2} + \frac{1}{x^4} = \left(\frac{x^2}{4} + \frac{1}{x^2}\right)^2.$$

The length of the graph over $[1, 4]$ is

$$L = \int_1^4 \sqrt{1 + [f'(x)]^2}\,dx = \int_1^4 \left(\frac{x^2}{4} + \frac{1}{x^2}\right)dx$$

$$= \left[\frac{x^3}{12} - \frac{1}{x}\right]_1^4 = \left(\frac{64}{12} - \frac{1}{4}\right) - \left(\frac{1}{12} - 1\right) = \frac{72}{12} = 6.$$

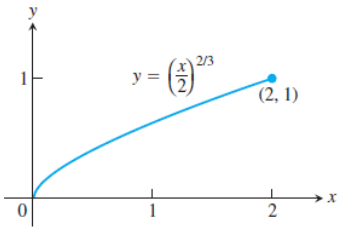


FIGURE The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$

EXAMPLE Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (3).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.26).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (4) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy \qquad \text{Eq. (4) with } c = 0, d = 1.$$

$$= \frac{1}{9}\cdot\frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1 \qquad \text{Let } u = 1 + 9y, du/9 = dy, \text{ integrate, and substitute back.}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

■

EXAMPLE Find the length of the curve

$$y = \frac{1}{2}(e^x + e^{-x}), \qquad 0 \le x \le 2.$$

Solution $a = 0, b = 2$, and

$$y = \frac{1}{2}(e^x + e^{-x})$$

$$\frac{dy}{dx} = \frac{1}{2}(e^x - e^{-x})$$

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{2x} - 2 + e^{-2x})$$

$$1 + \left(\frac{dy}{dx}\right)^2 = \frac{1}{4}(e^{2x} + 2 + e^{-2x}) = \left[\frac{1}{2}(e^x + e^{-x})\right]^2.$$

The length of the curve from $x = 0$ to $x = 2$ is

$$L = \int_0^2 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_0^2 \frac{1}{2}(e^x + e^{-x})\, dx \qquad \text{Eq. (3) with } a = 0, b = 2$$

$$= \frac{1}{2}\left[e^x - e^{-x}\right]_0^2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.63.$$

■

Solve Ex 6.3 Page386 Problem 1 to 17

5.1 Areas of Surfaces of Revolution

When you jump rope, the rope sweeps out a surface in the space around you similar to what is called a *surface of revolution*. The surface surrounds a volume of revolution, and many applications require that we know the area of the surface rather than the volume it encloses. In this section we define areas of surfaces of revolution. More general surfaces are treated later.

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx.$$

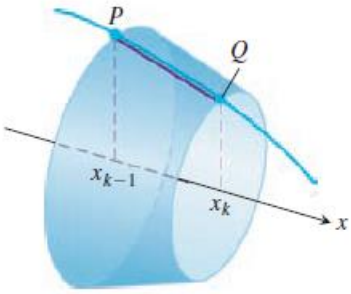


FIGURE The line segment joining P and Q sweeps out a frustum of a cone.

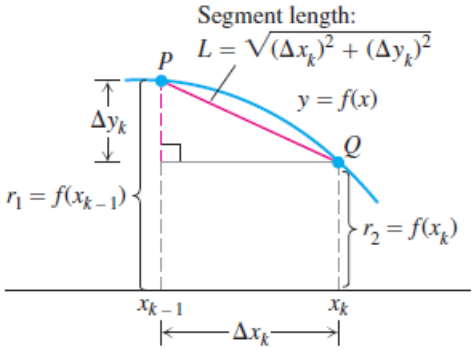


FIGURE Dimensions associated with the arc and line segment PQ.

EXAMPLE Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis

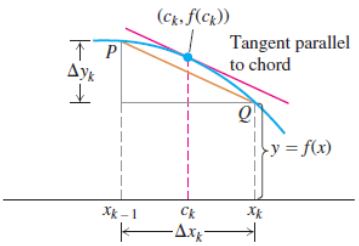
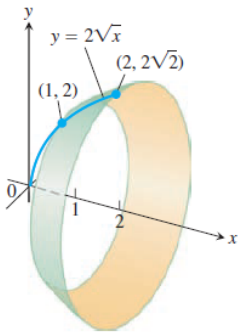


FIGURE 6.33 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.



Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}}.$$

First, we perform some algebraic manipulation on the radical in the integrand to transform it into an expression that is easier to integrate.

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2}$$

$$= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.$$

With these substitutions, we have

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\,\frac{\sqrt{x+1}}{\sqrt{x}}\, dx = 4\pi \int_1^2 \sqrt{x+1}\, dx$$

$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right). \qquad \blacksquare$$

Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y) \sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

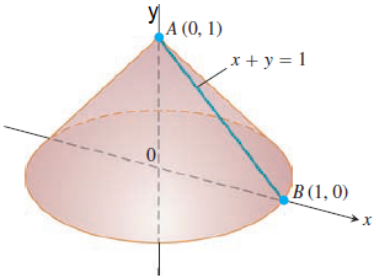


FIGURE Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE The line segment $x = 1 - y$, $0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.35. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \qquad d = 1, \qquad x = 1 - y, \qquad \frac{dx}{dy} = -1,$$

$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 2\pi(1 - y)\sqrt{2}\, dy$$
$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$
$$= \pi\sqrt{2}.$$

The results agree, as they should. ■

Solve EX6.4 Page391

Problem 1 to 21

6.Finding Volume by integration

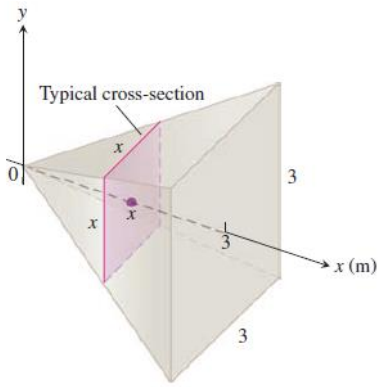
DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

Calculating the Volume of a Solid

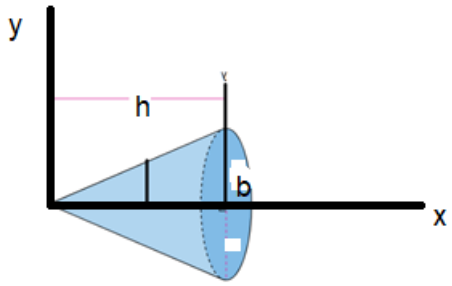
1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ to find the volume.

EXAMPLE 1 A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.



$$V = \int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = 9\ \mathrm{m}^3.$$

Example . Find the volume of the cone below by using integration

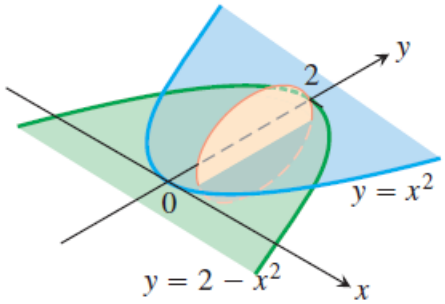


$$V = \int_0^h \pi y^2 dx$$

But $\frac{h}{b} = \frac{x}{y}$ then $V = \int_0^h \pi(\frac{b}{h})^2 x^2 dx = \pi \frac{b^2}{h^2}\frac{h^3}{3} = \pi b^2 \frac{h}{3}$

Example

The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.



$$A(x) = \frac{\pi(\text{diameter})^2}{4} = \frac{\pi[(2-x^2)-x^2]^2}{4} = \frac{\pi[2(1-x^2)]^2}{4} = \pi(1 - 2x^2 + x^4);\ a = -1,\ b = 1;$$

$$V = \int_a^b A(x)\,dx = \int_{-1}^1 \pi(1 - 2x^2 + x^4)\,dx = \pi\left[x - \frac{2}{3}x^3 + \frac{x^5}{5}\right]_{-1}^1 = 2\pi\left(1 - \frac{2}{3} + \frac{1}{5}\right) = \frac{16\pi}{15}$$

Solve Ex 6.1 Page 371

Problem 2,3,4,9,15 to 21