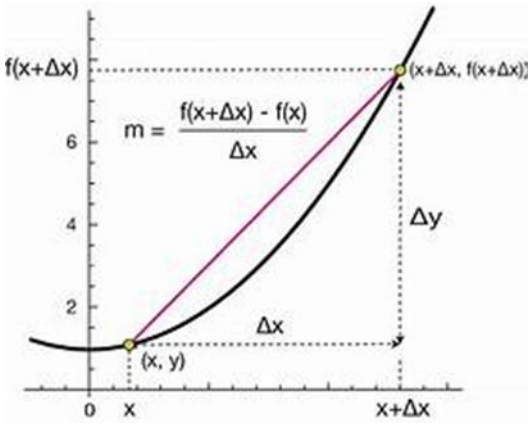
# DIFFERENTIATION

The derivative is one of the key ideas in calculus, and we use it to solve a wide range of problems involving tangents and rates of change.

## 1.1 Definition of differentiation

if $y = f(x)$ then
$y' = f'(x) = \frac{dy}{dx} = D_x$
is denoted the derivative and is defined as

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$




then $y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

Example . By definitions, find $\frac{dy}{dx}$ of the function $y = x + 1$ .

$$f(x) = x + 1$$
$$f(x + \Delta x) = (x + \Delta x) + 1$$

Using the relation $y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

then $y' = \lim_{\Delta x \to 0} \frac{x+\Delta x+1-(x+1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}$ =1

Ex2. By definitions, find $\frac{dy}{dx}$ of the function $y = x^2 + 2$

$$f(x) = x^2 + 1$$
$$f(x + \Delta x) = (x + \Delta x)^2 + 1$$

Using the relation

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + \Delta x^2 + 1 - (x^2+1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2x\Delta x + \Delta x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x(2x + \Delta x)}{\Delta x} = 2x$$

Home work

By definitions, find $\frac{dy}{dx}$ of the function

1) $y = \sqrt{x}$
2) $y = x^3 + 2$

## 1.2 Basic rule of Differentiation

### Derivative of a Constant Function

If $f$ has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

**Power Rule for Positive Integers:**

If $n$ is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

**EXAMPLE 1** Differentiate the following powers of $x$.

(a) $x^3$ (b) $x^{2/3}$ (c) $x^{\sqrt{2}}$ (d) $\frac{1}{x^4}$ (e) $x^{-4/3}$ (f) $\sqrt{x^{2+\pi}}$

**Solution**

(a) $\frac{d}{dx}(x^3) = 3x^{3-1} = 3x^2$ (b) $\frac{d}{dx}(x^{2/3}) = \frac{2}{3}x^{(2/3)-1} = \frac{2}{3}x^{-1/3}$

(c) $\frac{d}{dx}\left(x^{\sqrt{2}}\right) = \sqrt{2}x^{\sqrt{2}-1}$ (d) $\frac{d}{dx}\left(\frac{1}{x^4}\right) = \frac{d}{dx}(x^{-4}) = -4x^{-4-1} = -4x^{-5} = -\frac{4}{x^5}$

(e) $\frac{d}{dx}(x^{-4/3}) = -\frac{4}{3}x^{-(4/3)-1} = -\frac{4}{3}x^{-7/3}$

(f) $\frac{d}{dx}\left(\sqrt{x^{2+\pi}}\right) = \frac{d}{dx}\left(x^{1+(\pi/2)}\right) = \left(1 + \frac{\pi}{2}\right)x^{1+(\pi/2)-1} = \frac{1}{2}(2 + \pi)\sqrt{x^{\pi}}$

### Derivative Constant Multiple Rule

If $u$ is a differentiable function of $x$, and $c$ is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

### Derivative Sum Rule

If $u$ and $v$ are differentiable functions of $x$, then their sum $u + v$ is differentiable at every point where $u$ and $v$ are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

### Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

### Derivative Product Rule

If $u$ and $v$ are differentiable at $x$, then so is their product $uv$, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

### Derivative Quotient Rule

If $u$ and $v$ are differentiable at $x$ and if $v(x) \neq 0$, then the quotient $u/v$ is differentiable at $x$, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}.$$

## 1.3. Derivatives of Trigonometric Functions

**The derivative of the sine function is the cosine function:**

$$\frac{d}{dx}(\sin x) = \cos x.$$

**EXAMPLE 1** We find derivatives of the sine function involving differences, products, and quotients.

**(a)** $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$

$$= 2x - \cos x$$

**(b)** $y = e^x \sin x$:

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\sin x) + \frac{d}{dx}(e^x)\sin x \qquad \text{Product Rule}$$

$$= e^x \cos x + e^x \sin x$$

$$= e^x (\cos x + \sin x)$$

**(c)** $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \dfrac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$

$$= \frac{x \cos x - \sin x}{x^2}$$

■

**The derivative of the cosine function is the negative of the sine function:**

$$\frac{d}{dx}(\cos x) = -\sin x.$$

**EXAMPLE 2** We find derivatives of the cosine function in combinations with other functions.

**(a)** $y = 5e^x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5e^x) + \frac{d}{dx}(\cos x) \qquad \text{Sum Rule}$$
$$= 5e^x - \sin x$$

**(b)** $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) \qquad \text{Product Rule}$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x$$

**(c)** $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$
$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$
$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$
$$= \frac{1}{1 - \sin x}$$ ■

**The derivatives of the other trigonometric functions:**

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$
$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

**EXAMPLE** Find $d(\tan x)/dx$.

**Solution** We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$
$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$
$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

Home work : Find dy/dx for the following functions

**11.** $y = \dfrac{\cot x}{1 + \cot x}$ **12.** $y = \dfrac{\cos x}{1 + \sin x}$

**13.** $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$ **14.** $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$

**15.** $y = x^2 \sin x + 2x \cos x - 2 \sin x$

**16.** $y = x^2 \cos x - 2x \sin x - 2 \cos x$

**17.** $f(x) = x^3 \sin x \cos x$ **18.** $g(x) = (2 - x) \tan^2 x$

In Exercises 19–22, find $ds/dt$.

**19.** $s = \tan t - e^{-t}$ **20.** $s = t^2 - \sec t + 5e^t$

**21.** $s = \dfrac{1 + \csc t}{1 - \csc t}$ **22.** $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

**23.** $r = 4 - \theta^2 \sin \theta$ **24.** $r = \theta \sin \theta + \cos \theta$

**25.** $r = \sec \theta \csc \theta$ **26.** $r = (1 + \sec \theta) \sin \theta$

In Exercises 27–32, find $dp/dq$.

**27.** $p = 5 + \dfrac{1}{\cot q}$ **28.** $p = (1 + \csc q) \cos q$

**29.** $p = \dfrac{\sin q + \cos q}{\cos q}$ **30.** $p = \dfrac{\tan q}{1 + \tan q}$

**31.** $p = \dfrac{q \sin q}{q^2 - 1}$ **32.** $p = \dfrac{3q + \tan q}{q \sec q}$

**33.** Find $y''$ if

**a.** $y = \csc x$. **b.** $y = \sec x$.

**34.** Find $y^{(4)} = d^4 y/dx^4$ if

**a.** $y = -2 \sin x$. **b.** $y = 9 \cos x$.

in Exercise 35-37 find dy/dx

35. $y = \tan(3x^2)$ 36. $y = tanx\ sinx$

37. $y = 2\ sin\frac{x}{2} - x\ cos\frac{x}{2}$ 38. $y = sin\ x^3$

39. $y = x^2 \sin x$ 40. $y = \sqrt{sec(3x^2)}$

## 1.4 Derivative of the Natural Logarithm Function

if $a \neq 1 \; and \; x > 0 \; then \; y = \log_a x$ is equivalent to $a^y = x$

The natural log function can be defined by

$$y = \log_e x = \frac{lnx}{lne} = ln\, x \quad \text{Since} \quad \ln e = 1$$

Where $y = lnx$ is equivalent $x = e^y$

some properties of natural log.

$$\ln(xy) = lnx + lny$$
$$\ln\left(\frac{x}{y}\right) = lnx - lny$$
$$lnx^n = nlnx$$

Example

find y' for the following functions

1) $y = \log_5(x+1)^2$

$$y = \frac{2ln(x+1)}{ln5}$$

$$y' = \frac{2(\frac{1}{x+1})}{ln5} = \frac{2}{ln5(x+1)}$$

2) $y + lnxy = 1$

$$y + lnx + lny = 1$$
$$\frac{dy}{dx} + \frac{1}{x} + \frac{1}{y}\frac{dy}{dx} = 0$$
$$\frac{dy}{dx}\left(1 + \frac{1}{y}\right) = -\frac{1}{x}$$
$$y' = \frac{-y}{x(y+1)}$$

$$y = \ln x$$
$$e^y = x \qquad \text{Inverse function relationship}$$
$$\frac{d}{dx}(e^y) = \frac{d}{dx}(x) \qquad \text{Differentiate implicitly}$$
$$e^y \frac{dy}{dx} = 1 \qquad \text{Chain Rule}$$
$$\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}. \qquad e^y = x$$

No matter which derivation we use, the derivative of $y = \ln x$ with respect to $x$ is

$$\frac{d}{dx}(\ln x) = \frac{1}{x}, \quad x > 0.$$

The Chain Rule extends this formula for positive functions $u(x)$:

$$\frac{d}{dx}\ln u = \frac{d}{du}\ln u \cdot \frac{du}{dx}$$

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0.$$

## *The Derivative of $\log_a u$

If $u$ is a differentiable function of $x$ and $u > 0$, the Chain Rule gives the following formula.

For $a > 0$ and $a \neq 1$,

$$\frac{d}{dx}\log_a u = \frac{1}{u \ln a}\frac{du}{dx}. \qquad (7)$$

To find the derivative of $\log_a u$ for an arbitrary base(a $> 0$ , $a \neq 1$) , we start with the change-of-base formula for logarithms and express in terms of natural logarithms,

$$\log_a u = \frac{lnu}{lna} \quad \text{then}$$
$$\frac{d}{dx}(\log_a u) = \frac{d}{dx}\left(\frac{lnu}{lna}\right) = \frac{1}{\ln a}\frac{1}{u}\frac{du}{dx} = \frac{1}{u\ ln\ a}\frac{du}{dx}$$

## The Derivatives of $a^u$

We start with the equation $a^x = e^{\ln(a^x)} = e^{x \ln a}$,

$$\frac{d}{dx}a^x = \frac{d}{dx}e^{x \ln a} = e^{x \ln a} \cdot \frac{d}{dx}(x \ln a)$$
$$= a^x \ln a.$$

If $a > 0$ and $u$ is a differentiable function of $x$, then $a^u$ is a differentiable function of $x$ and

$$\frac{d}{dx}a^u = a^u \ln a \frac{du}{dx}.$$

**EXAMPLE**

**(a)** $\frac{d}{dx}3^x = 3^x \ln 3$

**(b)** $\frac{d}{dx}3^{-x} = 3^{-x}(\ln 3)\frac{d}{dx}(-x) = -3^{-x}\ln 3$

**(c)** $\frac{d}{dx}3^{\sin x} = 3^{\sin x}(\ln 3)\frac{d}{dx}(\sin x) = 3^{\sin x}(\ln 3)\cos x$

**EXAMPLE**

$$\frac{d}{dx}\ln 2x = \frac{1}{2x}\frac{d}{dx}(2x) = \frac{1}{2x}(2) = \frac{1}{x}, \quad x > 0$$

### Derivatives of Logarithms

In Exercises , find the derivative of $y$ with respect to $x$, $t$, or $\theta$, as appropriate.

13. $y = \ln(t^2)$
14. $y = \ln(t^{3/2})$
15. $y = \ln\frac{3}{x}$
16. $y = \ln\frac{10}{x}$
17. $y = \ln(\theta + 1)$
18. $y = \ln(2\theta + 2$
19. $y = \ln x^3$
20. $y = (\ln x)^3$
21. $y = t(\ln t)^2$
22. $y = t\sqrt{\ln t}$
23. $y = \frac{x^4}{4}\ln x - \frac{x^4}{16}$
24. $y = (x^2 \ln x)^4$
25. $y = \frac{\ln t}{t}$
26. $y = \frac{1 + \ln t}{t}$
27. $y = \frac{\ln x}{1 + \ln x}$
28. $y = \frac{x \ln x}{1 + \ln x}$

### Logarithmic Differentiation

In Exercises , use logarithmic differentiation to find the derivative of $y$ with respect to the given independent variable.

41. $y = \sqrt{x(x + 1)}$
42. $y = \sqrt{(x^2 + 1)(x - 1)^2}$
43. $y = \sqrt{\frac{t}{t + 1}}$
44. $y = \sqrt{\frac{1}{t(t + 1)}}$

## 1.5 The derivative of inverse function sem.2

we can find the derivative of $y = sin^{-1}x$ using implicit differentiation as follow :

$$x = siny$$

$$1=\cos y\, y'$$

$$y' = \frac{1}{cosy}$$

$$y' = \frac{1}{\sqrt{1-x^2}} \quad \text{where}$$

Assume $\cos^2 y+\sin^2 y=1$ then

$$\cos y=\sqrt{1 - sin^2}y=\sqrt{1 - x^2}$$

and so on for the other trigonometric function then

**TABLE** Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1} u)}{dx} = \dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

2. $\dfrac{d(\cos^{-1} u)}{dx} = -\dfrac{du/dx}{\sqrt{1-u^2}}, \quad |u| < 1$

3. $\dfrac{d(\tan^{-1} u)}{dx} = \dfrac{du/dx}{1+u^2}$

4. $\dfrac{d(\cot^{-1} u)}{dx} = -\dfrac{du/dx}{1+u^2}$

5. $\dfrac{d(\sec^{-1} u)}{dx} = \dfrac{du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

6. $\dfrac{d(\csc^{-1} u)}{dx} = \dfrac{-du/dx}{|u|\sqrt{u^2-1}}, \quad |u| > 1$

## 1.6 The hyperbolic function and their derivative

The hyperbolic functions are formed by taking combinations of the two exponential functions $e^x$ *and* $e^{-x}$ . The hyperbolic functions simplify many mathematical expressions and occur frequently in mathematical applications. In this section, we introduce these functions, and their derivatives.

### Definitions and Identities

sem 2

The hyperbolic sine and hyperbolic cosine functions are defined by the equations

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}.$$

We pronounce $\sinh x$ as "cinch $x$," rhyming with "pinch $x$," and $\cosh x$ as "kosh $x$," rhyming with "gosh $x$." From this basic pair, we define the hyperbolic tangent, cotangent, secant, and cosecant functions.

**TABLE 7.4** Identities for hyperbolic functions

$$\cosh^2 x - \sinh^2 x = 1$$

$$\sinh 2x = 2 \sinh x \cosh x$$

$$\cosh 2x = \cosh^2 x + \sinh^2 x$$

$$\cosh^2 x = \frac{\cosh 2x + 1}{2}$$

$$\sinh^2 x = \frac{\cosh 2x - 1}{2}$$

$$\tanh^2 x = 1 - \operatorname{sech}^2 x$$

$$\coth^2 x = 1 + \operatorname{csch}^2 x$$

**TABLE 7.5** Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

Example

Prove that $\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$

$$\frac{d}{dx}(\sinh u) = \frac{d}{dx}\left(\frac{e^u - e^{-u}}{2}\right)$$
$$= \frac{e^u\, du/dx + e^{-u}\, du/dx}{2}$$
$$= \cosh u \frac{du}{dx}.$$

## 1.6.1 Inverse Hyperbolic Functions and their derivatives

Sem. 2

**TABLE 7.8** Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \quad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \quad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}, \quad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = -\frac{1}{|u|\sqrt{1+u^2}}\frac{du}{dx}, \quad u \neq 0$$

**EXAMPLE** Show that if $u$ is a differentiable function of $x$ whose values are greater than 1, then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}.$$

Let $cosh^{-1}x = y$

Then $x = cosh\, y$ Differentiate with respect to x then

$1 = \sinh y \frac{dy}{dx}$

$\frac{dy}{dx} = \frac{1}{sinhy} = \frac{1}{\sinh(cosh^{-1}x)}$ but $cosh^2x - sinh^2x = 1$

$\frac{dy}{dx} = \frac{1}{sinhy} = \frac{1}{\sinh(cosh^{-1}x)} = \frac{1}{\sqrt{cosh^2(cosh^{-1}x)-1}} = \frac{1}{\sqrt{cosh^2y-1}} = \frac{1}{\sqrt{x^2-1}}$

then

$$\frac{d}{dx}(\cosh^{-1} u) = \frac{1}{\sqrt{u^2 - 1}}\frac{du}{dx}.$$

Homework:

Use the definitions of cosh $x$ and sinh $x$ to show that

$$\cosh^2 x - \sinh^2 x = 1.$$

**Finding Derivatives** sem.2

In Exercises 13–24, find the derivative of $y$ with respect to the appropriate variable.

**13.** $y = 6 \sinh \frac{x}{3}$

**14.** $y = \frac{1}{2}\sinh(2x + 1)$

**15.** $y = 2\sqrt{t}\tanh\sqrt{t}$

**16.** $y = t^2 \tanh\frac{1}{t}$

**17.** $y = \ln(\sinh z)$

**18.** $y = \ln(\cosh z)$

In Exercises find the derivative of $y$ with respect to the appropriate variable.

**25.** $y = \sinh^{-1}\sqrt{x}$

**26.** $y = \cosh^{-1} 2\sqrt{x + 1}$

**27.** $y = (1 - \theta)\tanh^{-1}\theta$

**28.** $y = (\theta^2 + 2\theta)\tanh^{-1}(\theta + 1)$

**29.** $y = (1 - t)\coth^{-1}\sqrt{t}$

**30.** $y = (1 - t^2)\coth^{-1} t$

## 1.7 Second- and Higher-Order Derivatives

if $y = f(x)$ then;

the first derivative is $y' = f'(x)$

the second derivative is $y'' = D_x^2 f(x) = \frac{d^2x}{dy^2} = f''(x)$

the third derivative is $y''' = D_x^3 f(x) = \frac{d^3x}{dy^3} = f'''(x)$

and so on for the n derivative is $y^n = D_x^n f(x) = \frac{d^nx}{dy^n} = f^n(x)$

Example

Find the third derivative of $y = x^3 - 3x^2 + 2$

Ans:

$y' = 3x^2 - 6x$

$y'' = 6x - 6$

$y''' = 6$

**Example**

Find the second derivative of the function at point (1,2)

$$y = \frac{1}{x} + \sqrt{x^3}$$

$$y' = \frac{-1}{x^2} + \frac{1}{2}(x^3)^{\frac{-1}{2}}(3x^2) = \frac{-1}{x^2} + \frac{3\sqrt{x}}{2}$$

$$y'' = \frac{-(-2)}{x^3} + \frac{3}{4}(x)^{\frac{-1}{2}} = \frac{2}{x^3} + \frac{3}{4\sqrt{x}}$$

$$y'' = \frac{2}{1} + \frac{3}{4} = \frac{11}{4}$$

### Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$
2. $y = x^2 + x + 8$
3. $s = 5t^3 - 3t^5$
4. $w = 3z^7 - 7z^3 + 21z^2$
5. $y = \frac{4x^3}{3} - x + 2e^x$
6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$
7. $w = 3z^{-2} - \frac{1}{z}$
8. $s = -2t^{-1} + \frac{4}{t^2}$
9. $y = 6x^2 - 10x - 5x^{-2}$
10. $y = 4 - 2x - x^{-3}$
11. $r = \frac{1}{3s^2} - \frac{5}{2s}$
12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find $y'$ **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$
14. $y = (2x + 3)(5x^2 - 4x)$
15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$
16. $y = (1 + x^2)(x^{3/4} - x^{-3})$

Find the derivatives of the functions in Exercises 17–40.

17. $y = \frac{2x + 5}{3x - 2}$
18. $z = \frac{4 - 3x}{3x^2 + x}$
19. $g(x) = \frac{x^2 - 4}{x + 0.5}$
20. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$
21. $v = (1 - t)(1 + t^2)^{-1}$
22. $w = (2x - 7)^{-1}(x + 5)$
23. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$
24. $u = \frac{5x + 1}{2\sqrt{x}}$
25. $v = \frac{1 + x - 4\sqrt{x}}{x}$
26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$
27. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$
28. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$
29. $y = 2e^{-x} + e^{3x}$
30. $y = \frac{x^2 + 3e^x}{2e^x - x}$
31. $y = x^3e^x$
32. $w = re^{-r}$
33. $y = x^{9/4} + e^{-2x}$
34. $y = x^{-3/5} + \pi^{3/2}$
35. $s = 2t^{3/2} + 3e^2$
36. $w = \frac{1}{z^{1.4}} + \frac{\pi}{\sqrt{z}}$
37. $y = \sqrt[7]{x^2} - x^e$
38. $y = \sqrt[3]{x^{9.6}} + 2e^{1.3}$
39. $r = \frac{e^s}{s}$
40. $r = e^{\theta}\left(\frac{1}{\theta^2} + \theta^{-\pi/2}\right)$

Find the derivatives of all orders of the functions in Exercises 41–44

41. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$
42. $y = \frac{x^5}{120}$
43. $y = (x - 1)(x^2 + 3x - 5)$
44. $y = (4x^3 + 3x)(2 - x)$

Find the first and second derivatives of the functions in Exercise 45–52.

45. $y = \frac{x^3 + 7}{x}$
46. $s = \frac{t^2 + 5t - 1}{t^2}$
47. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$
48. $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$
49. $w = \left(\frac{1 + 3z}{3z}\right)(3 - z)$
50. $p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$
51. $w = 3z^2e^{2z}$
52. $w = e^z(z - 1)(z^2 + 1)$

## 1.8 The chain rule

$$If\ y = g(t)\ \ and\ x = f(t)\ \ then$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

also $$If\ y = g(t)\ \ and\ t = f(x)\ \ then$$

$$\frac{dy}{dx} = \frac{dy}{dt}\ \ \frac{dt}{dx}$$

Example. Use Chain rule and find $\frac{dy}{dx}$ in term of x and y.

a) $y = \frac{t^2}{t^2+1}$ and $t = \sqrt{x+1}$

$$\frac{dy}{dt} = \frac{(t^2+1)(2t)-t^2(2t)}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$$

$$\frac{dt}{dx} = \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{\sqrt{x+1}}$$

$\frac{dy}{dx} = \frac{dy}{dt}\ \frac{dt}{dx} = \frac{2t}{(t^2+1)^2}\frac{1}{\sqrt{x+1}}$ but $t = \sqrt{x+1}$ then

$\frac{dy}{dx} = \frac{2\sqrt{x+1}}{(x+2)^2}\frac{1}{\sqrt{x+1}} = \frac{2}{(x+2)^2}$

b) $y = \frac{1}{t^2+1}$ and $x = \sqrt{4t-1}$

$$\frac{dy}{dt} = \frac{(t^2+1)(0)-2t}{(t^2+1)^2} = \frac{-2t}{(t^2+1)^2}$$

$$\frac{dx}{dt} = \frac{4}{2}(4t-1)^{-\frac{1}{2}} = \frac{2}{\sqrt{4t-1}}$$

$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} == \frac{-2t}{(t^2+1)^2}\frac{\sqrt{4t-1}}{2} = -y^2x\frac{x^2+1}{4} = \frac{-y^2x(x^2+1)}{4}$

If you want to find the value of the derivative at point (2,1) then put x=2, y=1 in the above equation(i.e $\frac{dy}{dx} = \frac{-1(2)(4+1)}{4} = \frac{-5}{2}$)

## Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

| | |
|---|---|
| **1.** $y = 6u - 9, \quad u = (1/2)x^4$ | **2.** $y = 2u^3, \quad u = 8x - 1$ |
| **3.** $y = \sin u, \quad u = 3x + 1$ | **4.** $y = \cos u, \quad u = -x/3$ |
| **5.** $y = \cos u, \quad u = \sin x$ | **6.** $y = \sin u, \quad u = x - \cos x$ |
| **7.** $y = \tan u, \quad u = 10x - 5$ | **8.** $y = -\sec u, \quad u = x^2 + 7x$ |

9.

Find $dy/dx$ if $y = x$ by using the Chain Rule with $y$ as a composite of

**a.** $y = (u/5) + 7$ and $u = 5x - 35$

**b.** $y = 1 + (1/u)$ and $u = 1/(x - 1)$.

Find $dy/dx$ if $y = x^{3/2}$ by using the Chain Rule with $y$ as a composite of

**a.** $y = u^3$ and $u = \sqrt{x}$

**b.** $y = \sqrt{u}$ and $u = x^3$.

10. Use Chain rule and find $\frac{dy}{dx}$ in term of x and y.

a) $y = \frac{(t-1)^2}{t^2+1}$ and $x = \frac{1}{t^2} - 1$ at t=2

$b)\ = 1 - \frac{1}{t}$ and $t = \frac{1}{1-x}$ at x=2

## 1.9. Implicit differentiation

If the differentiation function is algebraic combination of x and y , then if we want to differentiates the function simply differentiates two side with respect to x. such function are written as

$x^2 + y^3 - 6xy = 0$ , $x^2 + y^3 - 24 = 0$ , $x + y^3 = 0$

These equations define an *implicit* relation between the variables *x* and *y*.

Example Find $\frac{dy}{dx}$ $if\ y^2 = x$

ans:

$$2y\frac{dy}{dx} = 1$$
$$\frac{dy}{dx} = \frac{1}{2y} = \frac{1}{\pm\sqrt{x}}$$

It is more convenient in this type of differentiation to write $\frac{dy}{dx}$ as $y'$ .

Example

Find $dy/dx$ if $y^2 = x^2 + \sin xy$

Ans:

$$2yy' = 2x + \cos(xy)\,(\,y + xy')$$
$$2yy' = 2x + \cos(xy)\ y + \cos(xy)\,(xy')$$
$$2yy' - cos(xy)\,(xy') = 2x + y\cos(xy)$$
$$y'[2y - x\,cos(xy)] = 2x + y\cos(xy)$$
$$y' = \frac{2x + y\,cos(xy)}{2y - xcos(xy)}$$

if the question is to determine $\frac{dy}{dx}$ at point(0,-1) then $y' = \frac{2x0+(-1)cos(0x-1)}{2(-1)-0(\cos(0x-1)} = \frac{-1}{-2} = 0.5$

In Exercises 37-42, use implicit differentiation to find $dy/dx$ and then $d^2y/dx^2$.

**37.** $x^2 + y^2 = 1$ **38.** $x^{2/3} + y^{2/3} = 1$

**39.** $y^2 = x^2 + 2x$ **40.** $y^2 - 2x = 1 - 2y$

**41.** $2\sqrt{y} = x - y$ **42.** $xy + y^2 = 1$

**43.** If $x^3 + y^3 = 16$, find the value of $d^2y/dx^2$ at the point $(2, 2)$.

**44.** If $xy + y^2 = 1$, find the value of $d^2y/dx^2$ at the point $(0, -1)$.

In Exercises 1–12, find the first and second derivatives.

**1.** $y = -x^2 + 3$

**2.** $y = x^2 + x + 8$

**3.** $s = 5t^3 - 3t^5$

**4.** $w = 3z^7 - 7z^3 + 21z^2$

**5.** $y = \frac{4x^3}{3} - x$

**6.** $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$

**7.** $w = 3z^{-2} - \frac{1}{z}$

**8.** $s = -2t^{-1} + \frac{4}{t^2}$

**9.** $y = 6x^2 - 10x - 5x^{-2}$

**10.** $y = 4 - 2x - x^{-3}$

**11.** $r = \frac{1}{3s^2} - \frac{5}{2s}$

**12.** $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

Find the first and second derivatives of the functions in Exercises 31–38.

**31.** $y = \frac{x^3 + 7}{x}$

**32.** $s = \frac{t^2 + 5t - 1}{t^2}$

**33.** $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

**34.** $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

**35.** $w = \left(\frac{1 + 3z}{3z}\right)(3 - z)$

**36.** $w = (z + 1)(z - 1)(z^2 + 1)$

Use implicit differentiation to find $dy/dx$ in Exercises 19-32.

**19.** $x^2y + xy^2 = 6$

**20.** $x^3 + y^3 = 18xy$

**21.** $2xy + y^2 = x + y$

**22.** $x^3 - xy + y^3 = 1$

**23.** $x^2(x - y)^2 = x^2 - y^2$

**24.** $(3xy + 7)^2 = 6y$

**25.** $y^2 = \frac{x - 1}{x + 1}$

**26.** $x^2 = \frac{x - y}{x + y}$

**27.** $x = \tan y$

**28.** $xy = \cot(xy)$

**29.** $x + \tan(xy) = 0$

**30.** $x + \sin y = xy$

**31.** $y \sin\left(\frac{1}{y}\right) = 1 - xy$

**32.** $y^2 \cos\left(\frac{1}{y}\right) = 2x + 2y$

33. $x^2y^2 = x^2 + y^2$

34. $(x + y)^3 + (x - y)^3 = x^4 + y^4$

35. $\frac{(x-y)}{(x-2y)} = 2 \; at \; p(3,1)$

36. $xy + 2x - 5y = 2 \; at \; p(3,2)$

## Differentiating Implicitly

Use implicit differentiation to find $dy/dx$ in Exercises 1–16.

**1.** $x^2y + xy^2 = 6$

**2.** $x^3 + y^3 = 18xy$

**3.** $2xy + y^2 = x + y$

**4.** $x^3 - xy + y^3 = 1$

**5.** $x^2(x - y)^2 = x^2 - y^2$

**6.** $(3xy + 7)^2 = 6y$

**7.** $y^2 = \dfrac{x - 1}{x + 1}$

**8.** $x^3 = \dfrac{2x - y}{x + 3y}$

**9.** $x = \tan y$

**10.** $xy = \cot(xy)$

**11.** $x + \tan(xy) = 0$

**12.** $x^4 + \sin y = x^3y^2$

**13.** $y \sin\left(\dfrac{1}{y}\right) = 1 - xy$

**14.** $x \cos(2x + 3y) = y \sin x$

**15.** $e^{2x} = \sin(x + 3y)$

**16.** $e^{x^2y} = 2x + 2y$

Find $dr/d\theta$ in Exercises 17–20.

**17.** $\theta^{1/2} + r^{1/2} = 1$

**18.** $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$

**19.** $\sin(r\theta) = \dfrac{1}{2}$

**20.** $\cos r + \cot\theta = e^{r\theta}$

## Second Derivatives

In Exercises 21–26, use implicit differentiation to find $dy/dx$ and then $d^2y/dx^2$.

**21.** $x^2 + y^2 = 1$

**22.** $x^{2/3} + y^{2/3} = 1$

**23.** $y^2 = e^{x^2} + 2x$

**24.** $y^2 - 2x = 1 - 2y$

**25.** $2\sqrt{y} = x - y$

**26.** $xy + y^2 = 1$

**27.** If $x^3 + y^3 = 16$, find the value of $d^2y/dx^2$ at the point $(2, 2)$.

**28.** If $xy + y^2 = 1$, find the value of $d^2y/dx^2$ at the point $(0, -1)$.

In Exercises 29 and 30, find the slope of the curve at the given points.

**29.** $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

**30.** $(x^2 + y^2)^2 = (x - y)^2$ at $(1, 0)$ and $(1, -1)$

## Slopes, Tangents, and Normals

In Exercises 31–40, verify that the given point is on the curve and find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

**31.** $x^2 + xy - y^2 = 1, \quad (2, 3)$

**32.** $x^2 + y^2 = 25, \quad (3, -4)$

## 1.10 L'HOPITALS RULE

L'Hopitals rule uses derivatives to evaluate limits involving indeterminate forms .

if $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{0}{0}$ or $\lim_{x\to a}\frac{f(x)}{g(x)}=\frac{\infty}{\infty}$ , then

$\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{f\prime(x)}{g\prime(x)}$ , $\lim_{x\to a}\frac{f\prime\prime(x)}{g\prime\prime(x)}$ ,... $\lim_{x\to a}\frac{fn(x)}{gn(x)}$

### Example

Evaluate the following limit

1) $\lim_{x\to 0}\frac{sinx}{x}$

$\lim_{x\to a}\frac{sin0}{0}=\frac{0}{0}$

$\lim_{x\to a}\frac{f\prime(x)}{g\prime(x)}=\lim_{x\to a}\frac{cosx}{1}=\frac{cos0}{1}=1$

2) $\lim_{x\to\infty}\frac{5x^5+3x^2+5}{10x^5+3x^3+10}=\lim_{x\to\infty}\frac{25x^4+6x}{50x^4+9x^2}=\lim_{x\to\infty}\frac{100x^3+6}{200x^3+18x}=\lim_{x\to\infty}\frac{300x^2+6}{600x^2+18}=\lim_{x\to\infty}\frac{600x}{1200x}=$

$\lim_{x\to\infty}\frac{600}{1200}=\frac{1}{2}$

Home work

Find the limits using L'Hopitals Rule

1) $\lim_{n\to\infty}\frac{5x^2-3x}{7x^2+1}$

2) $\lim_{x\to 1}(\frac{x}{x-1}-\frac{1}{lnx})$

3) $\lim_{x\to 0}(\frac{2sinx-sin2x}{x-sinx})$