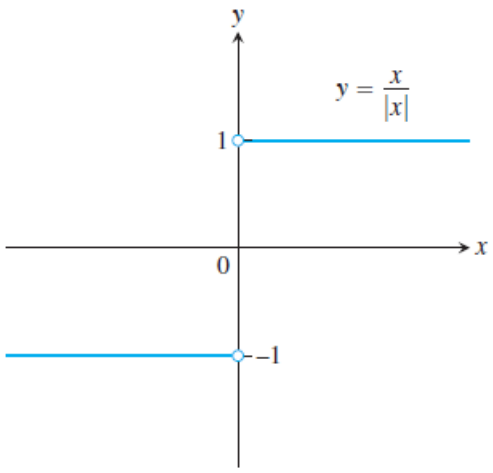
continuity

One-Sided Limits

One-sided limits, are limits as x approaches the number from the left-hand side (where x < c) or the right-hand side (x > c) only.

To have a limit L as x approaches c, a function f must be defined on *both sides* of c and its values $f(x)$ must approach L as x approaches c from either side. Because of this, ordinary limits are called **two-sided limits**. If f fails to have a two-sided limit at c, it may still have a one-sided limit, that is, a limit if the approach is only from one side. If the approach is from the right, the limit is a **right-hand limit**. From the left, it is a **left-hand limit**.



$$\lim_{x\to 0^+} f(x) = 1 \qquad \text{and} \qquad \lim_{x\to 0^-} f(x) = -1.$$

Then the function $y = \frac{x}{|x|}$ is not a two-sided limit at x=0 and it has a right-hand limit =1 and a left-hand limit =-1.

THEOREM A function $f(x)$ has a limit as x approaches c if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:

$$\lim_{x\to c} f(x) = L \quad \Leftrightarrow \quad \lim_{x\to c^-} f(x) = L \quad \text{and} \quad \lim_{x\to c^+} f(x) = L.$$

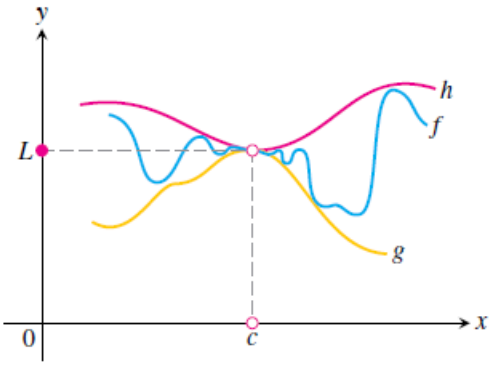
The Sandwich Theorem

The following theorem enables us to calculate a variety of limits. It is called the Sandwich Theorem because it refers to a function f whose values are sandwiched between the values of two other functions g and h that have the same limit L at a point c. Being trapped between the values of two functions that approach L, the values of f must also approach L

THEOREM —The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x \to c} g(x) = \lim_{x \to c} h(x) = L.$$

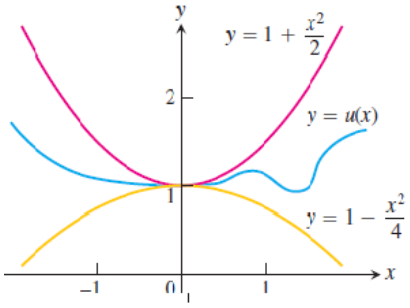
Then $\lim_{x \to c} f(x) = L$.



EXAMPLE Given that

$$1 - \frac{x^2}{4} \le u(x) \le 1 + \frac{x^2}{2} \quad \text{for all } x \ne 0,$$

find $\lim_{x \to 0} u(x)$, no matter how complicated u is.



Solution Since

$$\lim_{x \to 0}(1 - (x^2/4)) = 1 \quad \text{and} \quad \lim_{x \to 0}(1 + (x^2/2)) = 1,$$

the Sandwich Theorem implies that $\lim_{x \to 0} u(x) = 1$

Ex2.2. Page 75

Solve 63,64,65,66a.

Continuity at a Point

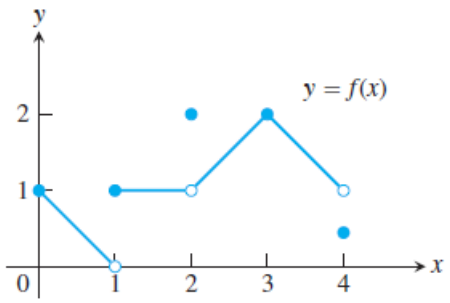
To understand continuity, it helps to consider a function like that in Figure 2.35, whose limits we investigated in Example 2 in the last section.

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

Example for the fig shown below, find the limits and test the continuity in x=0,1,2,3,4



EXAMPLE For the function graphed in Figure above

At $x = 0$: $\lim_{x\to 0^+} f(x) = 1$,

$\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0} f(x)$ do not exist. The function is not defined to the left of $x = 0$.

At $x = 1$: $\lim_{x\to 1^-} f(x) = 0$ even though $f(1) = 1$,

$\lim_{x\to 1^+} f(x) = 1$,

$\lim_{x\to 1} f(x)$ does not exist. The right- and left-hand limits are not equal.

At $x = 2$: $\lim_{x\to 2^-} f(x) = 1$,

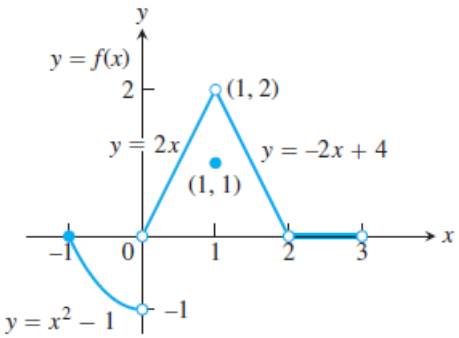
$\lim_{x\to 2^+} f(x) = 1$,

$\lim_{x\to 2} f(x) = 1$ even though $f(2) = 2$.

At $x = 3$: $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^+} f(x) = \lim_{x\to 3} f(x) = f(3) = 2$.

At $x = 4$: $\lim_{x\to 4^-} f(x) = 1$ even though $f(4) \neq 1$,

$\lim_{x\to 4^+} f(x)$ and $\lim_{x\to 4} f(x)$ do not exist. The function is not defined to the right of $x = 4$.



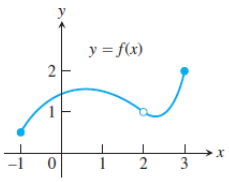
5. **a.** Does $f(-1)$ exist? No
   b. Does $\lim_{x\to -1^+} f(x)$ exist? yes
   c. Does $\lim_{x\to -1^+} f(x) = f(-1)$? yes
   d. Is f continuous at $x = -1$? No
6. **a.** Does $f(1)$ exist? yes
   b. Does $\lim_{x\to 1} f(x)$ exist? yes
   c. Does $\lim_{x\to 1} f(x) = f(1)$? No
   d. Is f continuous at $x = 1$? No
7. **a.** Is f defined at $x = 2$? (Look at the definition of f.) No
   b. Is f continuous at $x = 2$? No
8. At what values of x is f continuous? No
9. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$? f(2)=0
10. To what new value should $f(1)$ be changed to remove the discontinuity? f(1)=2

Solve the following questions

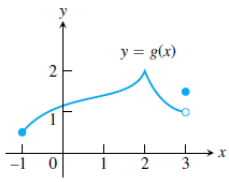
Continuity from Graphs

In Exercises 1–4, say whether the function graphed is continuous on $[-1, 3]$. If not, where does it fail to be continuous and why?

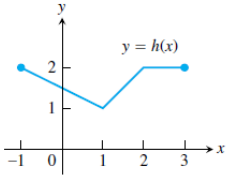
1.



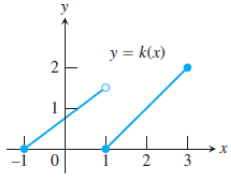
2.



3.



4.



Limits and Continuity

1. Graph the function

$$f(x) = \begin{cases} 1, & x \le -1 \\ -x, & -1 < x < 0 \\ 1, & x = 0 \\ -x, & 0 < x < 1 \\ 1, & x \ge 1. \end{cases}$$

Then discuss, in detail, limits, one-sided limits, continuity, and one-sided continuity of f at $x = -1, 0$, and 1. Are any of the discontinuities removable? Explain.

2. Repeat the instructions of Exercise 1 for

$$f(x) = \begin{cases} 0, & x \le -1 \\ 1/x, & 0 < |x| < 1 \\ 0, & x = 1 \\ 1, & x > 1. \end{cases}$$

At what points are the functions in Exercises 13–24 continuous?

13. $y = \dfrac{1}{x-2} - 3x$

14. $y = \dfrac{1}{(x+2)^2} + 4$

15. $y = \dfrac{x+1}{x^2 - 4x + 3}$

16. $y = \dfrac{x+3}{x^2 - 3x - 10}$

17. $y = |x - 1| + \sin x$

18. $y = \dfrac{1}{|x|+1} - \dfrac{x^2}{2}$

19. $y = \dfrac{\cos x}{x}$

20. $y = \dfrac{x+2}{\cos x}$

21. $y = \csc 2x$

22. $y = \tan \dfrac{\pi x}{2}$

23. $y = \dfrac{x \tan x}{x^2 + 1}$

24. $y = \dfrac{\sqrt{x^4 + 1}}{1 + \sin^2 x}$

For what values of a and b is

$$f(x) = \begin{cases} -2, & x \le -1 \\ ax - b, & -1 < x < 1 \\ 3, & x \ge 1 \end{cases}$$

continuous at every x?

For what values of a and b is

$$g(x) = \begin{cases} ax + 2b, & x \le 0 \\ x^2 + 3a - b, & 0 < x \le 2 \\ 3x - 5, & x > 2 \end{cases}$$

continuous at every x?