5.Integration by Partial fraction

So, let's do a quick review of partial fractions. We'll start with a rational expression in the form,

$$f(x)=\frac{P(x)}{Q(x)}$$

where both $P(x)$ and $Q(x)$ are polynomials and the degree of $P(x)$ is smaller than the degree of $Q(x)$. Recall that the degree of a polynomial is the largest exponent in the polynomial. Partial fractions can only be done if the degree of the numerator is strictly less than the degree of the denominator. That is important to remember.

Factor in denominator	Term in partial fraction decomposition
$ax+b$	$\frac{A}{ax+b}$
$(ax+b)^k$	$\frac{A_1}{ax+b}+\frac{A_2}{(ax+b)^2}+\cdots+\frac{A_k}{(ax+b)^k}, \quad k=1,2,3,\ldots$
ax^2+bx+c	$\frac{Ax+B}{ax^2+bx+c}$
$(ax^2+bx+c)^k$	$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_kx+B_k}{(ax^2+bx+c)^k}, \quad k=1,2,3,\ldots$

Example Evaluate the following integral.

$$\int \frac{3x+11}{x^2-x-6}dx$$

Solution

The first step is to factor the denominator as much as possible and get the form of the partial fraction decomposition. Doing this gives,

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A}{x-3}+\frac{B}{x+2}$$

The next step is to actually add the right side back up.

$$\frac{3x+11}{(x-3)(x+2)}=\frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Now, we need to choose *A* and *B* so that the numerators of these two are equal for every *x*. To do this we'll need to set the numerators equal.

$$3x+11=A(x+2)+B(x-3)$$

$$x=-2: \qquad 5=A(0)+B(-5) \quad \Rightarrow \quad B=-1$$
$$x=3 \quad :20=A(5)+B(0) \quad \Rightarrow \quad A=4$$

Then

$$\int \frac{3x+11}{x^2-x-6}dx = \int \frac{4}{x-3}-\frac{1}{x+2}dx$$
$$= \int \frac{4}{x-3}dx - \int \frac{1}{x+2}dx$$
$$= 4\ln|x-3| - \ln|x+2| + c$$

EXAMPLE Use partial fractions to evaluate

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}dx.$$

Solution The partial fraction decomposition has the form

$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$

To find the values of the undetermined coefficients A, B, and C, we clear fractions and get

$$x^2 + 4x + 1 = A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1)$$
$$= A(x^2 + 4x + 3) + B(x^2 + 2x - 3) + C(x^2 - 1)$$
$$= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C).$$

The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x, obtaining

$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

There are several ways of solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}dx = \int \left[\frac{3}{4}\frac{1}{x - 1} + \frac{1}{2}\frac{1}{x + 1} - \frac{1}{4}\frac{1}{x + 3}\right]dx$$
$$= \frac{3}{4}\ln|x - 1| + \frac{1}{2}\ln|x + 1| - \frac{1}{4}\ln|x + 3| + K,$$

EXAMPLE Use partial fractions to evaluate

$$\int \frac{6x + 7}{(x + 2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x + 7}{(x + 2)^2} = \frac{A}{x + 2} + \frac{B}{(x + 2)^2}$$

$$6x + 7 = A(x + 2) + B \qquad \text{Multiply both sides by } (x + 2)^2.$$

$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x + 7}{(x + 2)^2}\,dx = \int \left(\frac{6}{x + 2} - \frac{5}{(x + 2)^2}\right) dx$$

$$= 6\int \frac{dx}{x + 2} - 5\int (x + 2)^{-2}\,dx$$

$$= 6 \ln |x + 2| + 5(x + 2)^{-1} + C. \quad \blacksquare$$

EXAMPLE Use partial fractions to evaluate

$$\int \frac{dx}{x(x^2 + 1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2 + 1)^2} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} + \frac{Dx + E}{(x^2 + 1)^2}.$$

Multiplying by $x(x^2 + 1)^2$, we have

$$1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)x$$

$$= A(x^4 + 2x^2 + 1) + B(x^4 + x^2) + C(x^3 + x) + Dx^2 + Ex$$

$$= (A + B)x^4 + Cx^3 + (2A + B + D)x^2 + (C + E)x + A$$

If we equate coefficients, we get the system

$$A + B = 0, \quad C = 0, \quad 2A + B + D = 0, \quad C + E = 0, \quad A = 1.$$

Solving this system gives $A = 1, B = -1, C = 0, D = -1$, and $E = 0$. Thus,

$$\int \frac{dx}{x(x^2+1)^2} = \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx$$

$$= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2}$$

$$= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad u = x^2 + 1,\ du = 2x\,dx$$

$$= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K$$

$$= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \quad \blacksquare$$

Example . Evaluate the following integral.

$$\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx$$

Let's first get the general form of the partial fraction decomposition.

$$\frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}=\frac{A}{x-1}+\frac{Bx+C}{x^2+4}+\frac{Dx+E}{(x^2+4)^2}$$

Now, set numerators equal, expand the right side and collect like terms.

$$\begin{aligned}x^3+10x^2+3x+36 &= A(x^2+4)^2+(Bx+C)(x-1)(x^2+4)+(Dx+E)(x-1)\\ &=(A+B)x^4+(C-B)x^3+(8A+4B-C+D)x^2+\\ &\qquad(-4B+4C-D+E)x+16A-4C-E\end{aligned}$$

Setting coefficient equal gives the following system.

$$\left.\begin{array}{rr} x^4: & A+B=0\\ x^3: & C-B=1\\ x^2: & 8A+4B-C+D=10\\ x^1: & -4B+4C-D+E=3\\ x^0: & 16A-4C-E=36\end{array}\right\} \Rightarrow \quad A=2, B=-2, C=-1, D=1, E=0$$

Here's the integral.

$$\begin{aligned}\int \frac{x^3+10x^2+3x+36}{(x-1)(x^2+4)^2}\,dx &= \int \frac{2}{x-1}+\frac{-2x-1}{x^2+4}+\frac{x}{(x^2+4)^2}\,dx\\ &= \int \frac{2}{x-1}-\frac{2x}{x^2+4}-\frac{1}{x^2+4}+\frac{x}{(x^2+4)^2}\,dx\\ &= 2\ln|x-1|-\ln|x^2+4|-\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right)-\frac{1}{2}\frac{1}{x^2+4}+c\end{aligned}$$

Homework solve : Ex8.4 page 479

9,10,13,15,21,26,47