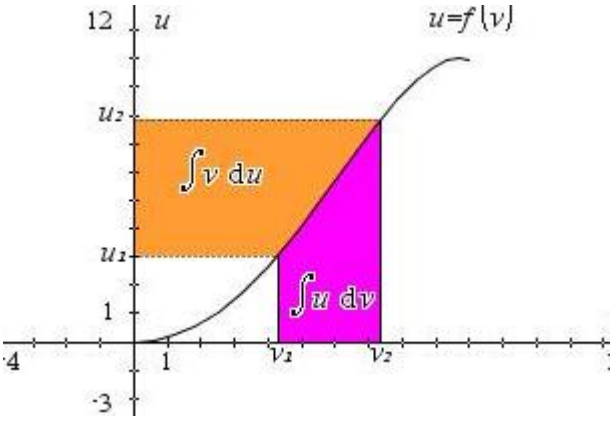
## Techniques of Integration

### 1.Integration by Parts

Integration by parts is used to integrate the product of two or more functions. The two functions are $u$ and $v$ . Among the two functions, the first function $u$ is selected such that its derivative formula exists, and the second function $v$ is chosen such that an integral of such a function exists. It is a technique for simplifying integrals of the form

$$\int u\,dv$$




then following the above drawing, it can be written as

$$\int u\,dv = uv - \int v du$$

**EXAMPLE 1** Find

$$\int x\cos x\,dx.$$

**Solution** We use the formula $\int u\,dv = uv - \int v\,du$ with

$$u = x, \qquad dv = \cos x\,dx,$$
$$du = dx, \qquad v = \sin x. \qquad \text{Simplest antiderivative of } \cos x$$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C. \qquad \blacksquare$$

**EXAMPLE 2** Find

$$\int \ln x \, dx.$$

**Solution** Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\quad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\quad v = x.$ Simplest antiderivative

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$

***Example* 3** Evaluate the following integral.

$$\int x\mathbf{e}^{6x} \, dx$$

Let

$$u = x \qquad dv = \mathbf{e}^{6x} \, dx$$

$$du = dx \qquad v = \int \mathbf{e}^{6x} \, dx = \frac{1}{6}\mathbf{e}^{6x}$$

The integral is then,

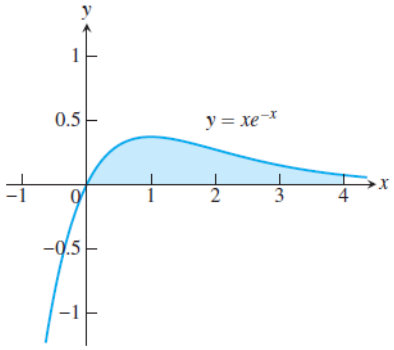
$$\int x\mathbf{e}^{6x} \, dx = \frac{x}{6}\mathbf{e}^{6x} - \int \frac{1}{6}\mathbf{e}^{6x} \, dx$$

$$= \frac{x}{6}\mathbf{e}^{6x} - \frac{1}{36}\mathbf{e}^{6x} + c$$

## 2. Integration by parts, define integral

$$\int_a^b u \, dv = uv\Big|_a^b - \int_a^b v \, du$$

**EXAMPLE 4** Find the area of the region bounded by the curve $y = xe^{-x}$ and the $x$-axis from $x = 0$ to $x = 4$.




**Solution** The region is shaded in Figure . Its area is

$$\int_0^4 xe^{-x}\,dx.$$

Let $u = x$, $dv = e^{-x}\,dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\,dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\,dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\,dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$

***Example*** **5** Evaluate the following integral.

$$\int_{-1}^{2} x\mathbf{e}^{6x}\,dx$$

***Solution***

This is the same integral that we looked at in the example so we'll use the same *u* and *dv* to get,

$$\begin{aligned}\int_{-1}^{2} x\mathbf{e}^{6x}\,dx &= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{6}\int_{-1}^{2}\mathbf{e}^{6x}\,dx \\ &= \frac{x}{6}\mathbf{e}^{6x}\Big|_{-1}^{2} - \frac{1}{36}\mathbf{e}^{6x}\Big|_{-1}^{2} \\ &= \frac{11}{36}\mathbf{e}^{12} + \frac{7}{36}\mathbf{e}^{-6}\end{aligned}$$

## 3. Tabular Integration

Tabular integration is a method of quickly integrating parts many times in sequence. This method requires that one of the functions in u.v be differentiable until it is zero. We must also be able to integrate the other function every time differentiate the first

function. To integrate with tabulation, create a table of 3 columns wide. In the first column insert the function u and its derivatives of a function till 0. In the second column, alternate the sign from (+) and (-) and so on. In the next column integrate every non-zero derivative. Now multiply the first cell in the table with the next two in the row below, Repeat this action for every row in the table. To get our integrated result, simply sum all of the terms together.

**EXAMPLE** Evaluate

$$\int x^2 e^x \, dx.$$

**Solution** With $\mathbf{u} = x^2$ and $\mathbf{dv} = e^x \, \mathbf{dx}$

| **u** and its derivatives | **sign** | **dv** and its integrals |
|---|---|---|
| $x^2$ | (+) | $e^x$ |
| $2x$ | (−) | $e^x$ |
| $2$ | (+) | $e^x$ |
| $0$ | | $e^x$ |

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2x e^x + 2e^x + C.$$

**EXAMPLE** Evaluate

$$\int x^3 \sin x \, dx.$$

**Solution** With $\mathbf{u} = x^3$ and $\mathbf{dv} = \sin x \, \mathbf{dx}$

| **u** and its derivatives | **sign** | **dv** and its integrals |
|---|---|---|
| $x^3$ | (+) | $\sin x$ |
| $3x^2$ | (−) | $-\cos x$ |
| $6x$ | (+) | $-\sin x$ |
| $6$ | (−) | $\cos x$ |
| $0$ | | $\sin x$ |

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C. \quad \blacksquare$$

Homework Solve , Ex 8.1 Page 459

5,8,26,33,35,45,46,47,52,67