4. Tangent half-angle substitution

The tangent half-angle substitution is a change of variables used for evaluating integrals, which converts a rational function of trigonometric functions of x into an ordinary rational function of t by setting t=tan(x/2). The general transformation formula is

$$\int f(sinx, \cos x)\, dx = \int f(t)\, dt$$

Introducing a new variable t= tan(x/2). sines and cosines can be expressed as rational functions of t and can be expressed as the product of dt and a rational function of t as follows:

$$\sin x = \frac{2t}{1+t^2}, \qquad \cos x = \frac{1-t^2}{1+t^2}, \qquad \text{and} \qquad dx = \frac{2}{1+t^2}\, dt.$$

Obtained from

$$\sin x = \frac{2\sin\frac{x}{2}\cos\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{2t}{1+t^2},$$

$$\cos x = \frac{\cos^2\frac{x}{2}-\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}+\sin^2\frac{x}{2}} = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}} = \frac{1-t^2}{1+t^2}.$$

Finally, since $t = \tan\frac{x}{2}$, differentiation rules imply

$$dt = \tfrac{1}{2}\left(1+\tan^2\tfrac{x}{2}\right)dx = \frac{1+t^2}{2}dx,$$

and thus

$$dx = \frac{2}{1+t^2}dt.$$

Ex 1

$$\begin{aligned}\int \csc x\,dx &= \int \frac{dx}{\sin x} \\ &= \int \left(\frac{1+t^2}{2t}\right)\left(\frac{2}{1+t^2}\,dt\right) \qquad t = \tan\frac{x}{2} \\ &= \int \frac{dt}{t} \\ &= \ln t + C \\ &= \ln\tan\frac{x}{2} + C.\end{aligned}$$

Ex2

$$\int \frac{dx}{1+\sin x} = \int \frac{\frac{2dt}{1+t^2}}{1+\frac{2t}{1+t^2}} = \int \frac{2dt}{1+t^2+2t} = \int \frac{2dt}{(t+1)^2} = -\frac{2}{t+1} + C = -\frac{2}{\tan\frac{x}{2}+1} + C.$$

Ex3

$$\int \frac{dx}{1+\cos\frac{x}{2}} = \int \frac{d\left(\frac{x}{2}\right)}{1+\cos\frac{x}{2}} = 2\int \frac{\frac{2dt}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} = 4\int \frac{dt}{1+\cancel{t^2}+1-\cancel{t^2}} = 2\int dt = 2t + C = 2\tan\frac{x}{4} + C$$

Ex4

$$\int \frac{dx}{4+5\cos\frac{x}{2}} = \int \frac{\frac{4dt}{1+t^2}}{4+5\cdot\frac{1-t^2}{1+t^2}} = \int \frac{4dt}{4(1+t^2)+5(1-t^2)} = 4\int \frac{dt}{4+4t^2+5-5t^2} = 4\int \frac{dt}{3^2-t^2} =$$

$$4\cdot\frac{1}{2\cdot 3}\ln\left|\frac{3+t}{3-t}\right| + C = \frac{2}{3}\ln\left|\frac{3+\tan\frac{x}{4}}{3-\tan\frac{x}{4}}\right| + C.$$

Homework Solve Prove that

$$\int \frac{dx}{3-5\sin x} = \frac{1}{4}\ln\left|\frac{\tan\frac{x}{2}-3}{3\tan\frac{x}{2}-1}\right| + C.$$

By using Tangent half-angle substitution